

# THE CO-HOPFIAN PROPERTY OF SURFACE BRAID GROUPS

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**ABSTRACT.** When both  $g$  and  $n$  are integers at least two, we give a description of any injective homomorphism from a finite index subgroup of the pure braid group with  $n$  strands on a closed orientable surface of genus  $g$ , into the pure braid group. As a consequence, we show that any finite index subgroup of the pure braid group is co-Hopfian.

## 1. INTRODUCTION

Let  $S$  be a connected, compact and orientable surface which may possibly have non-empty boundary. A description of any isomorphism between two finite index subgroups of the extended mapping class group  $\text{Mod}^*(S)$  of  $S$  is obtained in [17], [23] and [24]. A key step in these works is to compute the automorphism group of the complex of curves for  $S$ , denoted by  $\mathcal{C}(S)$ . More generally, to describe any injective homomorphism from a finite index subgroup of  $\text{Mod}^*(S)$  into  $\text{Mod}^*(S)$ , superinjectivity of a simplicial map from  $\mathcal{C}(S)$  into itself is introduced by Irmak [13] and is shown to be induced by an element of  $\text{Mod}^*(S)$  in [1], [3], [13], [14] and [15]. The same conclusion is obtained for injective simplicial maps from  $\mathcal{C}(S)$  into itself by Shackleton [28]. To prove similar results on the Torelli group and the Johnson kernel for a certain surface, variants of the complex of curves are introduced and studied in [6], [7], [8], [19], [20], [21] and [25].

Let  $\bar{S}$  be the closed surface obtained from  $S$  by attaching disks to all boundary components of  $S$ . We then have the homomorphism from the pure mapping class group  $\text{PMod}(S)$  of  $S$  onto the mapping class group  $\text{Mod}(\bar{S})$  of  $\bar{S}$ . We define  $P(S)$  to be the kernel of this homomorphism. If the genus of  $S$  is at least two, then  $P(S)$  is naturally identified with the fundamental group of the space of ordered distinct  $p$  points in  $\bar{S}$ , where  $p$  denotes the number of boundary components of  $S$ , as discussed in Theorem 4.2 of [5]. In our previous paper [22], we show that for some surfaces  $S$ , any isomorphism between two finite index subgroups of  $P(S)$  is the conjugation by an element of  $\text{Mod}^*(S)$ . The purpose of the present paper is to establish the same conclusion for any injective homomorphism from a finite index subgroup of  $P(S)$  into  $P(S)$ . As an immediate consequence of it, we show that any finite index subgroup  $\Gamma$  of  $P(S)$  is *co-Hopfian*, that is, any injective homomorphism from  $\Gamma$  into itself is surjective.

In [22], inspired by a work due to Irmak-Ivanov-McCarthy [16], we introduced the simplicial complex  $\mathcal{CP}(S)$  called the complex of HBCs and HBPs in  $S$ , on which  $\text{Mod}^*(S)$  naturally acts. We proved that any injective homomorphism from a finite index subgroup of  $P(S)$  into  $P(S)$  induces a superinjective map from  $\mathcal{CP}(S)$

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into itself. We also showed the natural isomorphism between  $\text{Mod}^*(S)$  and the automorphism group of  $\mathcal{CP}(S)$ . The present paper is thus devoted to showing that any superinjective map from  $\mathcal{CP}(S)$  into itself is surjective. The following theorem is a consequence of these results.

**Theorem 1.1.** *Let  $S$  be a connected, compact and orientable surface such that both the genus and the number of boundary components of  $S$  are at least two. Then any superinjective map from  $\mathcal{CP}(S)$  into itself is induced by an element of  $\text{Mod}^*(S)$ .*

As a by-product of the proof of this theorem, in Corollary 8.15, we also prove that any superinjective map from the subcomplex  $\mathcal{CP}_n(S)$  of  $\mathcal{CP}(S)$ , called the complex of HBCs and non-separating HBPs for  $S$ , into  $\mathcal{CP}(S)$  is induced by an element of  $\text{Mod}^*(S)$ . Combining the above theorem with Theorem 7.13 (i) in [22] and applying argument in Section 3 of [17], we obtain the following:

**Corollary 1.2.** *Let  $S$  be the surface in Theorem 1.1. Then for any finite index subgroup  $\Gamma$  of  $P(S)$  and any injective homomorphism  $f: \Gamma \rightarrow P(S)$ , there exists a unique element  $\gamma \in \text{Mod}^*(S)$  with  $f(x) = \gamma x \gamma^{-1}$  for any  $x \in \Gamma$ . In particular,  $\Gamma$  is co-Hopfian.*

Parallel results are proved for the subgroup  $P_s(S)$  of  $P(S)$  and the subcomplex  $\mathcal{CP}_s(S)$  of  $\mathcal{CP}(S)$ , called the complex of HBCs and separating HBPs for  $S$ , that are precisely defined in Section 2.2. Let us summarize the results on them.

**Theorem 1.3.** *Let  $S$  be the surface in Theorem 1.1. Then any superinjective map from  $\mathcal{CP}_s(S)$  into itself is induced by an element of  $\text{Mod}^*(S)$ .*

**Corollary 1.4.** *Let  $S$  be the surface in Theorem 1.1. Then for any finite index subgroup  $\Lambda$  of  $P_s(S)$  and any injective homomorphism  $h: \Lambda \rightarrow P_s(S)$ , there exists a unique element  $\lambda \in \text{Mod}^*(S)$  with  $h(y) = \lambda y \lambda^{-1}$  for any  $y \in \Lambda$ . In particular,  $\Lambda$  is co-Hopfian.*

This corollary is obtained by combining the last theorem with Theorem 7.13 (ii) in [22]. If the genus of a surface  $S$  is equal to zero, then we have the equalities  $P(S) = P_s(S) = \text{PMod}(S)$  and  $\mathcal{CP}(S) = \mathcal{CP}_s(S) = \mathcal{C}(S)$ . If the genus of  $S$  is equal to one, then we have the equalities  $P(S) = \mathcal{I}(S)$  and  $P_s(S) = \mathcal{K}(S)$ , where  $\mathcal{I}(S)$  is the Torelli group for  $S$  and  $\mathcal{K}(S)$  is the Johnson kernel for  $S$ . Moreover,  $\mathcal{CP}(S)$  and  $\mathcal{CP}_s(S)$  are equal to the Torelli complex for  $S$  and the complex of separating curves for  $S$ , respectively. We refer to [19] for a definition of these groups and complexes. It therefore follows from [3] and [20] that if  $S$  is a surface with the genus less than two and the Euler characteristic less than  $-2$ , then the same conclusions for  $S$  as in the above theorems hold. The co-Hopfian property of the braid groups on the disk, which are central extensions of the mapping class groups of holed spheres, is discussed in [2] and [4].

For a positive integer  $n$  and a manifold  $M$ , we define  $B_n(M)$  as the *braid group* of  $n$  strands on  $M$ , i.e., the fundamental group of the space of non-ordered distinct  $n$  points in  $M$ . We also define  $PB_n(M)$  as the *pure braid group* of  $n$  strands on  $M$ , i.e., the fundamental group of the space of ordered distinct  $n$  points in  $M$ . The group  $PB_n(M)$  is naturally identified with a subgroup of  $B_n(M)$  of index  $n!$ . If  $S$  is a surface of genus at least two with  $p$  boundary components, then the kernel of the homomorphism from  $\text{Mod}(S)$  onto  $\text{Mod}(\tilde{S})$  associated with the inclusion of  $S$  into  $\tilde{S}$  is identified with  $B_p(\tilde{S})$  (see Theorem 4.3 of [5]). Under this identification,

$P(S)$  is identified with  $PB_p(\bar{S})$ . As a consequence of Corollary 1.2, we obtain the following corollary, which answers to Question 4 in [2] affirmatively for closed orientable surfaces of genus at least two.

**Corollary 1.5.** *Let  $n$  be an integer at least two. Let  $M$  be a connected, closed and orientable surface of genus at least two. Then any finite index subgroup of  $B_n(M)$  is co-Hopfian.*

The proof of this corollary is presented in Section 2.5. In the next section, we introduce notation and terminology employed throughout the paper and recall basic properties of superinjective maps defined on  $\mathcal{CP}(S)$  etc. Afterward, we present an organization of the paper, outlining the proof of Theorems 1.1 and 1.3.

## 2. PRELIMINARIES

**2.1. Notation and terminology.** Unless otherwise stated, we always assume that a surface is connected, compact and orientable. Let  $S = S_{g,p}$  be a surface of genus  $g$  with  $p$  boundary components, and let  $\partial S$  denote the boundary of  $S$ . A simple closed curve in  $S$  is said to be *essential* in  $S$  if it is neither homotopic to a single point of  $S$  nor isotopic to a boundary component of  $S$ . Let  $V(S)$  denote the set of isotopy classes of essential simple closed curves in  $S$ . When there is no confusion, we mean by a curve in  $S$  either an essential simple closed curve in  $S$  or its isotopy class. An essential simple closed curve  $\alpha$  in  $S$  is said to be *non-separating* in  $S$  if  $S \setminus \alpha$  is connected, and otherwise  $\alpha$  is said to be *separating* in  $S$ . These properties depend only on the isotopy class of  $\alpha$ . For a separating curve  $\alpha$  in  $S$  and two components  $\partial_1, \partial_2$  of  $\partial S$ , we say that  $\alpha$  *separates*  $\partial_1$  and  $\partial_2$  if  $\partial_1$  and  $\partial_2$  are contained in distinct components of  $S \setminus \alpha$ .

Let  $i: V(S) \times V(S) \rightarrow \mathbb{Z}_{\geq 0}$  denote the *geometric intersection number*, i.e., the minimal cardinality of the intersection of representatives for two elements of  $V(S)$ . If  $a$  and  $b$  are essential simple closed curves in  $S$  with  $|a \cap b| = i([a], [b])$ , where  $[a]$  and  $[b]$  denote the isotopy classes of  $a$  and  $b$ , respectively, then we say that  $a$  and  $b$  *intersect minimally*. Let  $\Sigma(S)$  denote the set of non-empty finite subsets  $\sigma$  of  $V(S)$  with  $i(\alpha, \beta) = 0$  for any two elements  $\alpha, \beta \in \sigma$ . We extend  $i$  to the symmetric function on  $(V(S) \sqcup \Sigma(S))^2$  so that  $i(\alpha, \sigma) = \sum_{\beta \in \sigma} i(\alpha, \beta)$  and  $i(\sigma, \tau) = \sum_{\beta \in \sigma, \gamma \in \tau} i(\beta, \gamma)$  for any  $\alpha \in V(S)$  and  $\sigma, \tau \in \Sigma(S)$ . We say that two elements  $\sigma, \tau$  of  $V(S) \sqcup \Sigma(S)$  are *disjoint* if  $i(\sigma, \tau) = 0$ , and otherwise we say that they *intersect*. We say that two elements  $\alpha, \beta$  of  $V(S)$  *fill*  $S$  if there exists no element of  $V(S)$  disjoint from both  $\alpha$  and  $\beta$ .

For each  $\sigma \in \Sigma(S)$ , we denote by  $S_\sigma$  the surface obtained by cutting  $S$  along all curves in  $\sigma$ . When  $\sigma$  consists of a single curve  $\alpha$ , we denote it by  $S_\alpha$  for simplicity. We often identify a component of  $S_\sigma$  with a complementary component of a tubular neighborhood of a one-dimensional submanifold representing  $\sigma$  in  $S$  if there is no confusion. If  $Q$  is a component of  $S_\sigma$ , then  $V(Q)$  is naturally identified with a subset of  $V(S)$ .

Suppose that the boundary  $\partial S$  of  $S$  is non-empty. We say that a simple arc  $l$  in  $S$  is *essential* in  $S$  if

- $\partial l$  consists of two distinct points of  $\partial S$ ;
- $l$  meets  $\partial S$  only at its end points; and
- $l$  is not isotopic relative to  $\partial l$  to an arc in  $\partial S$ .

Unless otherwise stated, isotopy of essential simple arcs in  $S$  may move their end points, keeping them staying in  $\partial S$ . An essential simple arc  $l$  in  $S$  is said to be *separating* in  $S$  if  $S \setminus l$  is not connected. Otherwise  $l$  is said to be *non-separating* in  $S$ . These properties depend only on the isotopy class of  $l$ . Let  $\partial_1$  and  $\partial_2$  be distinct components of  $\partial S$ . We say that an essential simple arc  $l$  in  $S$  *connects*  $\partial_1$  and  $\partial_2$  if one of the end point of  $l$  lies in  $\partial_1$  and another in  $\partial_2$ .

**2.2. Complexes and groups associated to surfaces.** The following simplicial complex was introduced by Harvey [11].

**Complex  $\mathcal{C}(S)$ .** This is defined as the abstract simplicial complex such that the sets of vertices and simplices of  $\mathcal{C}(S)$  are  $V(S)$  and  $\Sigma(S)$ , respectively. The complex  $\mathcal{C}(S)$  is called the *complex of curves* for  $S$ .

We denote by  $\bar{S}$  the closed surface obtained from  $S$  by attaching disks to all boundary components of  $S$ . Let  $\mathcal{C}^*(\bar{S})$  be the simplicial cone over  $\mathcal{C}(\bar{S})$  with its cone point  $*$ . Namely,  $\mathcal{C}^*(\bar{S})$  is the abstract simplicial complex such that the set of vertices is the disjoint union  $V(\bar{S}) \sqcup \{*\}$ ; and the set of simplices is

$$\Sigma(\bar{S}) \cup \{ \sigma \cup \{*\} \mid \sigma \in \Sigma(\bar{S}) \cup \{\emptyset\} \}.$$

We then have the simplicial map

$$\pi: \mathcal{C}(S) \rightarrow \mathcal{C}^*(\bar{S})$$

associated with the inclusion of  $S$  into  $\bar{S}$ . Note that  $\pi^{-1}(*)$  consists of all separating curves in  $S$  cutting off a holed sphere from  $S$ .

**Hole-bounding curves (HBC).** A curve  $\alpha$  in  $S$  is called a *hole-bounding curve* (HBC) in  $S$  if  $\alpha$  lies in  $\pi^{-1}(*)$ . When the genus of  $S$  is positive and the holed sphere cut off by  $\alpha$  contains exactly  $k$  components of  $\partial S$ , we call  $\alpha$  a *k-HBC* in  $S$ . Note that we have  $2 \leq k \leq p$ .

If  $\alpha$  is a *k-HBC* in  $S$  and  $\partial_1, \dots, \partial_k$  are the components of  $\partial S$  contained in the holed sphere cut off by  $\alpha$ , then we say that  $\alpha$  *encircles*  $\partial_1, \dots, \partial_k$ .

Let  $\alpha$  be a 2-HBC in  $S$  encircling two components  $\partial_1, \partial_2$  of  $\partial S$ . Up to isotopy, there exists a unique essential simple arc in  $S$  connecting  $\partial_1$  with  $\partial_2$  and disjoint from  $\alpha$ . This arc is called the *defining arc* of  $\alpha$ . Conversely, if we have  $g \geq 1$  and  $p \geq 2$  and if  $l$  is an essential simple arc in  $S$  connecting two distinct components  $\partial_1, \partial_2$  of  $\partial S$ , then the boundary component of a regular neighborhood of the union  $l \cup \partial_1 \cup \partial_2$  is a 2-HBC in  $S$ . The 2-HBC is then called the curve in  $S$  *defined by*  $l$ .

**Hole-bounding pairs (HBP).** A pair  $\{\alpha, \beta\}$  of curves in  $S$  is called a *hole-bounding pair* (HBP) in  $S$  if  $\{\alpha, \beta\}$  is an edge of  $\mathcal{C}(S)$  and we have  $\pi(\alpha) = \pi(\beta) \neq *$ . We note that there exists a unique component of  $S_{\{\alpha, \beta\}}$  of genus zero if the genus of  $S$  is at least two. In this case, if that component contains exactly  $k$  components of  $\partial S$ , then we call the pair  $\{\alpha, \beta\}$  a *k-HBP* in  $S$ . Note that we have  $1 \leq k \leq p$ .

An HBP in  $S$  is said to be *non-separating* if both curves in it are non-separating in  $S$ , and it is said to be *separating* if both curves in it are separating in  $S$ . Note that each HBP in  $S$  is either separating or non-separating. Two disjoint HBPs  $a, b$  in  $S$  are said to be *equivalent* in  $S$  if  $\pi(a) = \pi(b)$ . Two disjoint curves  $\alpha, \beta$  in  $S$  are said to be *HBP-equivalent* in  $S$  if  $\pi(\alpha) = \pi(\beta)$ .

**Complexes  $\mathcal{CP}(S)$ ,  $\mathcal{CP}_n(S)$  and  $\mathcal{CP}_s(S)$ .** Let  $V_c(S)$  be the set of all isotopy classes of HBCs in  $S$ , and let  $V_p(S)$  be the set of all isotopy classes of HBPs in  $S$ .

We define  $\mathcal{CP}(S)$  to be the abstract simplicial complex such that the set of vertices is the disjoint union  $V_c(S) \sqcup V_p(S)$ ; and a non-empty finite subset  $\sigma$  of  $V_c(S) \sqcup V_p(S)$  is a simplex of  $\mathcal{CP}(S)$  if and only if any two elements of  $\sigma$  are disjoint.

We denote by  $V_{np}(S)$  and  $V_{sp}(S)$  the subsets of  $V_p(S)$  consisting of all non-separating HBPs and all separating HBPs, respectively. We define  $\mathcal{CP}_n(S)$  and  $\mathcal{CP}_s(S)$  as the full subcomplexes of  $\mathcal{CP}(S)$  spanned by  $V_c(S) \sqcup V_{np}(S)$  and  $V_c(S) \sqcup V_{sp}(S)$ , respectively.

**Mapping class groups.** The *extended mapping class group*  $\text{Mod}^*(S)$  for  $S$  is the group of isotopy classes of homeomorphisms from  $S$  onto itself, where isotopy may move points in  $\partial S$ . The *mapping class group*  $\text{Mod}(S)$  for  $S$  is the group of isotopy classes of orientation-preserving homeomorphisms from  $S$  onto itself. The *pure mapping class group*  $\text{PMod}(S)$  for  $S$  is the group of isotopy classes of orientation-preserving homeomorphisms from  $S$  onto itself that fix each component of  $\partial S$  as a set. Both  $\text{Mod}(S)$  and  $\text{PMod}(S)$  are finite index subgroups of  $\text{Mod}^*(S)$ . For each  $\alpha \in V(S)$ , we denote by  $t_\alpha \in \text{PMod}(S)$  the (*left*) *Dehn twist* about  $\alpha$ .

**Surface braid groups.** We have the surjective homomorphism

$$\iota: \text{PMod}(S) \rightarrow \text{Mod}(\bar{S})$$

associated with the inclusion of  $S$  into  $\bar{S}$ . We define  $P(S)$  to be  $\ker \iota$ . By the Birman exact sequence,  $P(S)$  is generated by all elements of the forms  $t_\alpha$  and  $t_\beta t_\gamma^{-1}$  with  $\alpha \in V_c(S)$  and  $\{\beta, \gamma\} \in V_p(S)$  (see Section 4.1 in [5]). We define  $P_s(S)$  as the group generated by all elements of the forms  $t_\alpha$  and  $t_\beta t_\gamma^{-1}$  with  $\alpha \in V_c(S)$  and  $\{\beta, \gamma\} \in V_{sp}(S)$ . The group  $P_s(S)$  is of infinite index in  $P(S)$  because no non-zero power of  $t_\delta t_\epsilon^{-1}$  with  $\{\delta, \epsilon\} \in V_{np}(S)$  belongs to  $P_s(S)$  by Lemma 2.3 in [22].

**2.3. Superinjective maps.** We review basic properties of superinjective maps defined between simplicial complexes introduced in the previous subsection.

**Definition 2.1.** Let  $S$  be a surface, and let  $X$  and  $Y$  be any of the simplicial complexes,  $\mathcal{C}(S)$ ,  $\mathcal{CP}(S)$ ,  $\mathcal{CP}_n(S)$  and  $\mathcal{CP}_s(S)$ . We denote by  $V(X)$  and  $V(Y)$  the sets of vertices of  $X$  and  $Y$ , respectively. We mean by a *superinjective map*  $\phi: X \rightarrow Y$  a simplicial map  $\phi: X \rightarrow Y$  satisfying  $i(\phi(a), \phi(b)) \neq 0$  for any two vertices  $a, b \in V(X)$  with  $i(a, b) \neq 0$ .

Note that a map  $\phi: V(X) \rightarrow V(Y)$  defines a simplicial map from  $X$  into  $Y$  if and only if  $i(\phi(a), \phi(b)) = 0$  for any two vertices  $a, b \in V(X)$  with  $i(a, b) = 0$ . It can be checked that any superinjective map from  $X$  into  $Y$  is injective, and that for any superinjective map  $\phi: X \rightarrow Y$ , if the induced map from  $V(X)$  into  $V(Y)$  is surjective, then  $\phi$  is a simplicial isomorphism from  $X$  onto  $Y$ .

**Lemma 2.2.** Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  and  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}(S)$  be superinjective maps. Then  $\phi$  sends each non-separating HBP to an HBP, and  $\psi$  sends each separating HBP to an HBP.

This lemma is a direct consequence of the following description of simplices of maximal dimension in  $\mathcal{CP}(S)$ .

**Proposition 2.3** ([22, Proposition 3.5]). Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then we have

$$\dim(\mathcal{CP}(S)) = \dim(\mathcal{CP}_s(S)) = \dim(\mathcal{CP}_n(S)) = \binom{p+1}{2} - 1.$$

Moreover, for any simplex  $\sigma$  of  $\mathcal{CP}(S)$  of maximal dimension, there exists a unique simplex  $s = \{\beta_0, \beta_1, \dots, \beta_p\}$  of  $\mathcal{C}(S)$  such that

- any two curves in  $s$  are HBP-equivalent; and
- $\sigma$  consists of all HBPs of two curves in  $s$ .

Using this description of simplices of maximal dimension, we can also show that the maps  $\phi$  and  $\psi$  in Lemma 2.2 preserve rooted simplices defined as follows.

**Definition 2.4.** Let  $S$  be a surface, and let  $\sigma$  be a simplex of  $\mathcal{CP}(S)$  consisting of HBPs. We say that  $\sigma$  is *rooted* if there exists a curve  $\alpha$  in  $S$  contained in any HBP of  $\sigma$ . In this case, if  $|\sigma| \geq 2$ , then  $\alpha$  is uniquely determined and called the *root curve* of  $\sigma$ .

Rooted simplices are first introduced in [19] for the Torelli complex of  $S$  in an analogous way. The proof of the following two lemmas are verbatim translations of those of the cited lemmas, where only superinjective maps from  $\mathcal{CP}(S)$  into itself and ones from  $\mathcal{CP}_s(S)$  into itself are dealt with.

**Lemma 2.5** ([22, Lemma 3.12]). *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  preserves rooted simplices. Moreover, the same conclusion holds for any superinjective map from  $\mathcal{CP}_s(S)$  into  $\mathcal{CP}(S)$ .*

**Lemma 2.6** ([22, Lemma 3.13]). *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. Pick a simplex  $\sigma$  of  $\mathcal{CP}_n(S)$  of maximal dimension, and let  $\{\alpha_0, \alpha_1, \dots, \alpha_p\}$  denote the collection of curves in HBPs of  $\sigma$ . Then there exists a collection of curves in  $S$ ,  $\{\beta_0, \beta_1, \dots, \beta_p\}$ , satisfying the equality*

$$\phi(\{\alpha_j, \alpha_k\}) = \{\beta_j, \beta_k\}$$

*for any distinct  $j, k = 0, 1, \dots, p$ . Moreover, the same conclusion holds for any superinjective map from  $\mathcal{CP}_s(S)$  into  $\mathcal{CP}(S)$  and any simplex of  $\mathcal{CP}_s(S)$  of maximal dimension.*

**2.4. Strategy.** Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Our purpose is to show surjectivity of any superinjective map from  $\mathcal{CP}(S)$  into itself and any superinjective map from  $\mathcal{CP}_s(S)$  into itself.

In Section 3, we discuss topological properties of vertices preserved by a superinjective map  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$ . Specifically, we show that for any integers  $j$  and  $k$  with  $2 \leq j \leq p$  and  $1 \leq k \leq p$ , the map  $\phi$  preserves  $j$ -HBCs and non-separating  $k$ -HBPs, respectively. In particular, the inclusion  $\phi(\mathcal{CP}_n(S)) \subset \mathcal{CP}_n(S)$  is proved. The proof of these facts in the cases  $p = 2$  and  $p \geq 3$  are fairly different. In the former case, the pentagons in  $\mathcal{CP}_n(S)$  described in Figures 1 and 3 play a central role. The proof in the latter case is more straightforward.

In Section 4, we study a superinjective map  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  and show that  $\psi$  preserves  $j$ -HBCs and  $k$ -HBPs, respectively, for any  $j$  and  $k$ . The proof in the cases  $p = 2$  and  $p \geq 3$  are different as in Section 3. In the former case, the hexagons in  $\mathcal{CP}_s(S)$  described in Figures 4 and 6 are considered.

In Section 5, we focus on the case  $p = 2$  and discuss topological types of hexagons in  $\mathcal{CP}_n(S)$ , provided that topological types of their vertices are given. As a result, for any superinjective map  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  and any non-separating 2-HBP  $a$  in  $S$ , the map from the link of  $a$  into the link of  $\phi(a)$ , defined as the restriction of  $\phi$ ,

induces an injective simplicial map between the Farey graphs associated with the holed spheres cut off by  $a$  and  $\phi(a)$  from  $S$ . We then conclude that it is surjective and that  $\phi$  sends the link of  $a$  onto the link of  $\phi(a)$ .

In Section 6, we still focus on the case  $p = 2$  and obtain a similar result for any superinjective map  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  and any separating 2-HBP in  $S$  by studying topological types of hexagons in  $\mathcal{CP}_s(S)$ .

In Section 7, we introduce two natural subcomplexes of the complex of arcs, denoted by  $\mathcal{D}(X, \partial)$  and  $\mathcal{D}(Y)$ , and show that any injective simplicial maps from those complexes into themselves are surjective. In the case  $p = 2$ , properties of hexagons proved in Sections 5 and 6 are also used to show that the restrictions of superinjective maps  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  and  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  to the link of any 1-HBP induce injective simplicial maps between those complexes.

In Section 8, we prove surjectivity of any superinjective maps  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  and  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  by induction on  $p$ . Surjectivity of any superinjective map from  $\mathcal{CP}(S)$  into itself then follows.

**2.5. Proof of Corollary 1.5.** Assuming Corollary 1.2, we prove Corollary 1.5. Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . We denote by  $j: \text{Mod}(S) \rightarrow \text{Mod}(\tilde{S})$  the homomorphism associated with the inclusion of  $S$  into  $\tilde{S}$ , which is an extension of  $\iota$ . We define  $B(S)$  as the kernel of  $j$ . Our aim is to prove that any finite index subgroup of  $B(S)$  is co-Hopfian.

Let  $\Gamma$  be a finite index subgroup of  $B(S)$ . Let  $f: \Gamma \rightarrow \Gamma$  be an injective homomorphism. We put  $\Gamma_0 = P(S) \cap f^{-1}(P(S) \cap \Gamma)$ , which is a finite index subgroup of  $P(S)$ . By Corollary 1.2, there exists  $\gamma \in \text{Mod}^*(S)$  with  $f(x) = \gamma x \gamma^{-1}$  for any  $x \in \Gamma_0$ . For any  $y \in \Gamma$  and any HBC  $\alpha$  in  $S$ , there exists a non-zero integer  $N$  such that  $t_\alpha^N$  belongs to  $\Gamma_0$ . We then have

$$f(y t_\alpha^N y^{-1}) = f(y) f(t_\alpha^N) f(y)^{-1} = f(y) \gamma t_\alpha^N \gamma^{-1} f(y)^{-1} = t_{f(y)\gamma\alpha}^N.$$

On the other hand, we have

$$f(y t_\alpha^N y^{-1}) = f(t_{y\alpha}^N) = \gamma t_{y\alpha}^N \gamma^{-1} = t_{\gamma y\alpha}^N.$$

We thus have  $f(y)\gamma\alpha = \gamma y\alpha$  for any HBC  $\alpha$  in  $S$ . Replacing  $\alpha$  with any HBP  $b = \{b_1, b_2\}$  in  $S$  and replacing  $t_\alpha$  with  $t_{b_1} t_{b_2}^{-1}$ , we can show the equality  $f(y)\gamma b = \gamma y b$  along a verbatim argument. Since the action of  $\text{Mod}^*(S)$  on  $\mathcal{CP}(S)$  is faithful by Lemma 2.2 in [22], the equality  $f(y) = \gamma y \gamma^{-1}$  holds.

We have the equality  $\gamma^{-1} B(S) \gamma = B(S)$  because  $B(S)$  is a normal subgroup of  $\text{Mod}^*(S)$ . It follows that

$$[B(S) : \Gamma] \leq [B(S) : f(\Gamma)] = [B(S) : \gamma \Gamma \gamma^{-1}] = [B(S) : \Gamma].$$

The equality  $[B(S) : \Gamma] = [B(S) : f(\Gamma)]$  holds. We thus have  $f(\Gamma) = \Gamma$ . Corollary 1.5 is proved.

### 3. SUPERINJECTIVE MAPS FROM $\mathcal{CP}_n(S)$ INTO $\mathcal{CP}(S)$

We mean by a *pentagon* in  $\mathcal{CP}(S)$  a subgraph of  $\mathcal{CP}(S)$  consisting of five vertices  $v_1, \dots, v_5$  with  $i(v_k, v_{k+1}) = 0$  and  $i(v_k, v_{k+2}) \neq 0$  for each  $k \bmod 5$ . In this case, let us say that the *pentagon* is defined by the 5-tuple  $(v_1, \dots, v_5)$ . A *pentagon* in  $\mathcal{CP}_n(S)$  is defined as a *pentagon* in  $\mathcal{CP}(S)$  each of whose vertices belongs to  $\mathcal{CP}_n(S)$ . Examples of *pentagons* in  $\mathcal{CP}_n(S_{g,2})$  are described in Figures 1 and 3.

The following two lemmas are observations on pentagons in  $\mathcal{CP}(S_{g,2})$  consisting of HBPs.

**Lemma 3.1.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Then there exists no pentagon  $\Pi$  in  $\mathcal{CP}(S)$  such that*

- *any vertex of  $\Pi$  corresponds to an HBP; and*
- *there exists a curve in  $S$  contained in any HBP of  $\Pi$ .*

*Proof.* Assuming that there exists such a  $\Pi$  defined by a 5-tuple  $(a, b, c, d, e)$ , we deduce a contradiction. Let  $\alpha$  denote the curve shared by the five HBPs of  $\Pi$ . Label all boundary components of  $S_\alpha$  by  $\partial_1, \partial_2, \partial_3$  and  $\partial_4$  so that  $\partial_1$  and  $\partial_2$  correspond to  $\alpha$ , and  $\partial_3$  and  $\partial_4$  are components of  $\partial S$ . We define  $a^1$  as the curve in  $a$  distinct from  $\alpha$  and define  $b^1, c^1, d^1$  and  $e^1$  similarly.

We can deduce a contradiction if all vertices of  $\Pi$  are assumed to correspond to a 1-HBP, by considering the component of  $\partial S$  contained in the pair of pants cut off by a 1-HBP. We may thus assume that  $a$  is a 2-HBP and that  $a^1$  encircles  $\partial_1, \partial_3$  and  $\partial_4$  as a curve in  $S_\alpha$  if  $\alpha$  is non-separating, or in a component of  $S_\alpha$  if  $\alpha$  is separating. It follows that  $b$  and  $e$  are 1-HBPs. Without loss of generality, we may assume that  $b^1$  encircles  $\partial_1$  and  $\partial_3$ . If both  $c$  and  $d$  were 1-HBPs, then  $c^1$  would encircle  $\partial_2$  and  $\partial_4$ , and  $d^1$  would encircle  $\partial_1$  and  $\partial_3$ . It turns out that  $e^1$  encircles  $\partial_2$  and  $\partial_4$  and cannot be disjoint from  $a^1$ . This is a contradiction. If  $c$  were a 2-HBP, then  $c^1$  would have to encircle  $\partial_1, \partial_3$  and  $\partial_4$ . It follows that each of  $d^1$  and  $e^1$  encircles either  $\partial_1$  and  $\partial_3$  or  $\partial_1$  and  $\partial_4$  and that  $d^1$  and  $e^1$  cannot be disjoint and distinct. This is also a contradiction. In an analogous way, one can deduce a contradiction if  $d$  is assumed to be a 2-HBP.  $\square$

**Lemma 3.2.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Let  $(a, b_1, c_1, c_2, b_2)$  be a 5-tuple defining a pentagon  $\Pi$  in  $\mathcal{CP}(S)$  such that*

- *any vertex of  $\Pi$  corresponds to an HBP; and*
- *any of the edges  $\{a, b_1\}, \{a, b_2\}$  and  $\{c_1, c_2\}$  is rooted.*

*Let  $\alpha_1, \alpha_2$  and  $\beta$  denote the root curves of the three edges in the second condition, respectively. Then the equality  $\alpha_1 = \alpha_2$  holds, and  $\alpha_1$  and  $\beta$  are disjoint, but not HBP-equivalent.*

*Proof.* We define curves  $b_1^1, b_2^1, c_1^1$  and  $c_2^1$  so that we have

$$b_1 = \{\alpha_1, b_1^1\}, \quad b_2 = \{\alpha_2, b_2^1\}, \quad c_1 = \{\beta, c_1^1\}, \quad c_2 = \{\beta, c_2^1\}.$$

It follows from  $i(b_1, c_1) = i(b_2, c_2) = 0$  that  $i(\alpha_1, \beta) = i(\alpha_2, \beta) = 0$ . Let  $Q$  denote the holed sphere cut off by  $a$  from  $S$ .

If  $\alpha_1$  and  $\alpha_2$  were distinct, then we would have the equality  $a = \{\alpha_1, \alpha_2\}$ . The conditions  $i(c_1, a) \neq 0$  and  $i(c_2, a) \neq 0$  imply  $i(c_1^1, \alpha_2) \neq 0$  and  $i(c_2^1, \alpha_1) \neq 0$ . Suppose that  $a$  is a 2-HBP. It then follows that  $b_1^1$  and  $b_2^1$  are curves in  $Q$  separating the two boundary components of  $Q$  that correspond to  $\alpha_1$  and  $\alpha_2$ . Since we have  $i(c_1^1, \alpha_2) \neq 0$  and  $i(c_1^1, b_1^1) = i(c_1^1, \alpha_1) = 0$ , the intersection  $c_1^1 \cap Q$  consists of mutually isotopic, essential simple arcs in  $Q$ . Since  $i(c_2^1, \alpha_1) \neq 0$  and  $i(c_2^1, b_2^1) = i(c_2^1, \alpha_2) = 0$ , the intersection  $c_2^1 \cap Q$  also consists of mutually isotopic, essential simple arcs in  $Q$ . It however follows from  $i(c_1, c_2) = 0$  that the equality  $b_1^1 = b_2^1$  has to hold because  $b_1^1$  is a boundary component of a regular neighborhood of the union  $(c_1^1 \cap Q) \cup \alpha_2$  and a similar property holds for  $b_2^1$ . This contradicts  $i(b_1, b_2) \neq 0$ . We next suppose that  $a$  is a 1-HBP. The component  $Q$  is then a pair of pants containing  $\alpha_1$  and  $\alpha_2$



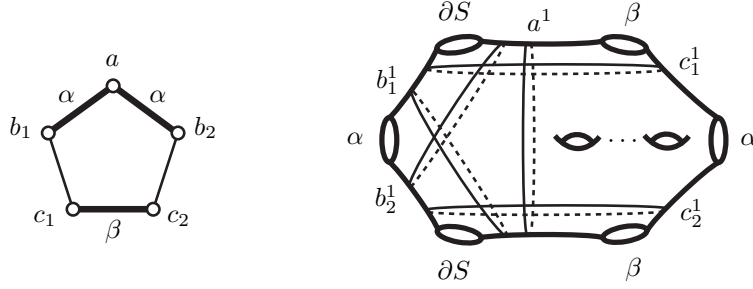


FIGURE 1. A pentagon in  $\mathcal{CP}_n(S_{g,2})$  with  $a = \{\alpha, a^1\}$ ,  $b_j = \{\alpha, b_j^1\}$  and  $c_j = \{\beta, c_j^1\}$  for each  $j = 1, 2$ . Thick edges of the pentagon in the left hand side and nearby symbols signify rooted edges and the root curves of them, respectively. Rooted edges and root curves of them are similarly indicated in other Figures. The surface obtained by cutting  $S_{g,2}$  along  $\alpha$  and  $\beta$  is drawn in the right hand side.

as boundary components. Since we have  $i(c_1^1, \alpha_2) \neq 0$ ,  $i(c_2^1, \alpha_1) \neq 0$  and  $i(c_1^1, \alpha_1) = i(c_2^1, \alpha_2) = 0$ , any component of  $c_1^1 \cap Q$  and that of  $c_2^1 \cap Q$  have to intersect. This contradicts  $i(c_1, c_2) = 0$ .

We have shown the equality  $\alpha_1 = \alpha_2$  and denote the curve by  $\alpha$ . We now prove the latter assertion of the lemma. By Lemma 3.1, we have  $\alpha \neq \beta$ . Assuming that  $\alpha$  and  $\beta$  are HBP-equivalent, we deduce a contradiction. It then follows that  $b_1$  and  $c_1$  are equivalent and have a common curve  $\gamma$ . If  $\gamma = \alpha$ , then we have  $c_1 = \{\alpha, \beta\}$ , and this contradicts  $i(c_1, b_2) \neq 0$ . Similarly, if  $\gamma = \beta$ , then we have  $b_1 = \{\alpha, \beta\}$ , and this contradicts  $i(b_1, c_2) \neq 0$ . We thus have  $\gamma \neq \alpha$  and  $\gamma \neq \beta$ . This implies the equalities  $b_1 = \{\alpha, \gamma\}$  and  $c_1 = \{\beta, \gamma\}$ . It follows from  $i(c_2, \alpha) = i(c_2, \gamma) = 0$  that we have  $i(c_2, b_1) = 0$ . This is a contradiction.  $\square$

As an application of the last two lemmas, we prove the following property on a superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  in the case  $p = 2$ . Subsequently, we prove an analogous property in the case  $p \geq 3$ .

**Lemma 3.3.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. Let  $a$  and  $b$  be non-separating 1-HBPs in  $S$  such that*

- *$a$  and  $b$  are disjoint; and*
- *if we choose a curve  $\alpha$  in  $a$  and a curve  $\beta$  in  $b$ , then  $S_{\{\alpha, \beta\}}$  is connected.*

*Then the HBPs  $\phi(a)$  and  $\phi(b)$  are not equivalent.*

*Proof.* Using Figure 1, we can find a pentagon  $\Pi$  in  $\mathcal{CP}_n(S)$  defined by a 5-tuple  $(x, y_1, z_1, z_2, y_2)$  such that

- $y_1 = a$  and  $z_1 = b$ ;
- any vertex of  $\Pi$  corresponds to an HBP; and
- any of the edges  $\{x, y_1\}$ ,  $\{x, y_2\}$  and  $\{z_1, z_2\}$  is rooted.

Applying Lemma 3.2 to  $\phi(\Pi)$ , we obtain the lemma.  $\square$

**Lemma 3.4.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 3$ . Then the following assertions hold:*

- (i) Let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. If  $a$  and  $b$  are non-separating HBPs in  $S$  which are disjoint, but not equivalent, then the HBPs  $\phi(a)$  and  $\phi(b)$  are not equivalent.
- (ii) Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. If  $a$  and  $b$  are separating HBPs in  $S$  which are disjoint, but not equivalent, then the HBPs  $\psi(a)$  and  $\psi(b)$  are not equivalent.

*Proof.* We prove assertion (i). Replacing  $a$  and  $b$  by HBPs equivalent to themselves, we may assume that  $a$  is a 1-HBP and that there exists a simplex  $s = \{\beta_1, \dots, \beta_p\}$  of  $\mathcal{C}(S)$  such that

- any curve in  $s$  is disjoint from  $a$ ;
- any two curves in  $s$  are HBP-equivalent; and
- $b$  is an HBP of two curves in  $s$ .

Let  $\sigma$  denote the simplex of  $\mathcal{CP}_n(S)$  consisting of all HBPs of two curves in  $s$ . For each  $j = 1, \dots, p$ , we define  $\sigma_j$  to be the rooted subsimplex of  $\sigma$  that consists of  $p-1$  HBPs and whose root curve is equal to  $\beta_j$ . Let  $\gamma_j$  denote the root curve of  $\phi(\sigma_j)$ . It follows from Lemma 2.6 that the equality

$$\phi(\{\beta_j, \beta_k\}) = \{\gamma_j, \gamma_k\}$$

holds for any distinct  $j, k = 1, \dots, p$ .

If  $\phi(a)$  were equivalent to  $\phi(b)$ , then those two HBPs would be contained in a simplex of  $\mathcal{CP}(S)$  of maximal dimension. Proposition 2.3 shows that for each  $j = 1, \dots, p$ , there exists an HBP  $b_j$  in  $\sigma_j$  such that  $\phi(a)$  and  $\phi(b_j)$  share a curve. If this curve were not equal to  $\gamma_j$ , then the inclusion  $\phi(b_j) \subset \phi(a) \cup \phi(b'_j)$  would hold for any HBP  $b'_j$  in  $\sigma_j$  distinct from  $b_j$ , which exists because of  $p \geq 3$ . This contradicts the existence of an HBP  $c$  with  $i(c, a) = i(c, b'_j) = 0$  and  $i(c, b_j) \neq 0$ . It thus turns out that for each  $j = 1, \dots, p$ ,  $\phi(a)$  and  $\phi(b_j)$  share  $\gamma_j$ . This is a contradiction because  $\phi(a)$  consists of two curves and we have  $p \geq 3$ .

A verbatim proof shows assertion (ii).  $\square$

In the rest of this section, we discuss topological properties of vertices preserved by any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$ .

**Lemma 3.5.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  preserves non-separating HBPs.*

*Proof.* The following argument appears in the proof of Lemma 3.14 in [22]. Pick a simplex  $\sigma$  of maximal dimension in  $\mathcal{CP}_n(S)$  which consists of HBPs of two curves in the collection  $s = \{\alpha_0, \alpha_1, \dots, \alpha_p\}$  of non-separating curves described in Figure 2. Let  $\beta_1, \beta_2$  and  $\beta_3$  be the curves in  $S$  described in Figure 2. Note that  $\beta_2$  is not HBP-equivalent to any curve of  $s$  and that  $S_{\{\alpha_j, \beta_2\}}$  is connected for each  $j = 0, \dots, p$ . By Lemma 2.6, there exist curves  $\gamma_j$  and  $\delta_k$  in  $S$  with

$$\phi(\{\alpha_{j_1}, \alpha_{j_2}\}) = \{\gamma_{j_1}, \gamma_{j_2}\}, \quad \phi(\{\beta_{k_1}, \beta_{k_2}\}) = \{\delta_{k_1}, \delta_{k_2}\}$$

for any distinct  $j_1, j_2 = 0, \dots, p$  and any distinct  $k_1, k_2 = 1, 2, 3$ . We put  $t = \{\gamma_0, \dots, \gamma_p\}$ . Since  $\{\delta_1, \delta_2\}$  intersects  $\gamma_0$  and is disjoint from any curve in  $t \setminus \{\gamma_0\}$  and since  $\{\delta_2, \delta_3\}$  intersects  $\gamma_p$  and is disjoint from any curve in  $t \setminus \{\gamma_p\}$ , we see that  $\delta_2$  is disjoint from any curve in  $t$ . By Lemmas 3.3 and 3.4,  $\delta_2$  is not HBP-equivalent to any curve of  $t$ . It follows that  $\{\delta_2, \delta_1\}$  is a 1-HBP in  $S$  and that there exists a  $(p-1)$ -HBP of two curves in  $t \setminus \{\gamma_0\}$  such that the other curves in it are contained in the holed sphere cut off by that  $(p-1)$ -HBP from  $S$ . Similarly,  $\{\delta_2, \delta_3\}$  is a

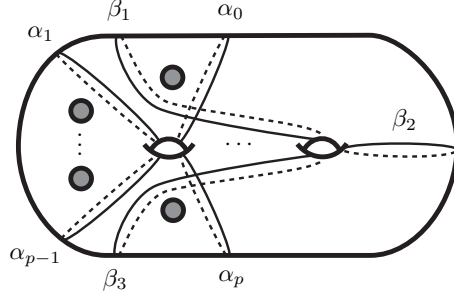


FIGURE 2.

1-HBP in  $S$ , and there exists a  $(p-1)$ -HBP of two curves in  $t \setminus \{\gamma_p\}$  such that the other curves in it are contained in the holed sphere cut off by that  $(p-1)$ -HBP from  $S$ . It thus follows that  $\{\gamma_0, \gamma_p\}$  is a  $p$ -HBP in  $S$ .

We now assume that any curve of  $t$  is separating in  $S$ . If  $\delta_2$  lies in the component of  $S_{\{\gamma_0, \gamma_p\}}$  of positive genus that contains  $\gamma_0$  as a boundary component, then the HBP  $\{\delta_2, \delta_3\}$  cannot intersect  $\gamma_p$ , kept disjoint from any curve in  $t \setminus \{\gamma_p\}$ . This is a contradiction. In a similar way, we can deduce a contradiction if we assume that  $\delta_2$  lies in the component of  $S_{\{\gamma_0, \gamma_p\}}$  of positive genus that contains  $\gamma_p$  as a boundary component. We thus proved that any curve of  $t$  is non-separating in  $S$ .  $\square$

**Lemma 3.6.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then for each integer  $k$  with  $1 \leq k \leq p$ , any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  preserves non-separating  $k$ -HBPs.*

*Proof.* Pick a non-separating 1-HBP  $a$ . There exists a simplex  $s = \{\beta_1, \dots, \beta_p\}$  of  $\mathcal{C}(S)$  such that

- any curve in  $s$  is disjoint from  $a$ ;
- any two curves in  $s$  are HBP-equivalent; and
- the surface obtained by cutting  $S$  along  $\beta_1$  and a curve of  $a$  is connected.

Let  $\sigma$  denote the simplex of  $\mathcal{CP}_n(S)$  consisting of all HBPs of two curves in  $s$ . It then follows that  $\phi(a)$  is a 1-HBP since  $\phi(a)$  and any HBP in  $\phi(\sigma)$  is not equivalent by Lemmas 3.3 and 3.4 and since  $\phi(\sigma)$  consists of all HBPs of two curves in a collection of  $p$  curves which are mutually HBP-equivalent.

We have shown that  $\phi$  preserves non-separating 1-HBPs. Note that for each non-separating  $k$ -HBP  $c$ , there exist curves  $\gamma_0, \dots, \gamma_k$  such that we have the equality  $c = \{\gamma_0, \gamma_k\}$  and for each  $j$ ,  $\{\gamma_j, \gamma_{j+1}\}$  is a 1-HBP. It follows that  $\phi(\{\gamma_j, \gamma_{j+1}\})$  is a 1-HBP. Using Lemma 2.6, we see that  $\phi(c)$  is a  $k$ -HBP.  $\square$

**Lemma 3.7.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then for each integer  $j$  with  $2 \leq j \leq p$ , any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  preserves  $j$ -HBCs.*

We first show this lemma when  $p = 2$ , using the following observation on root curves of edges of pentagons.

**Lemma 3.8** ([22, Lemma 4.6]). *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ , and let  $(a, b_1, c_1, c_2, b_2)$  be a 5-tuple defining a pentagon in  $\mathcal{CP}_n(S)$  such that*

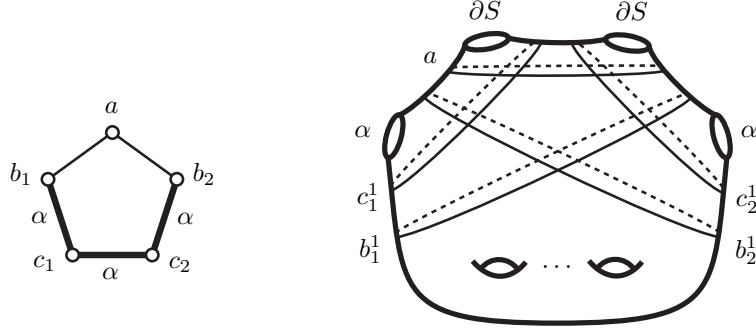


FIGURE 3. A pentagon in  $\mathcal{CP}_n(S_{g,2})$  with  $b_j = \{\alpha, b_j^1\}$  and  $c_j = \{\alpha, c_j^1\}$  for each  $j = 1, 2$ .

- $b_1$  and  $b_2$  are 2-HBPs;  $c_1$  and  $c_2$  are 1-HBPs; and
- each of the three edges  $\{b_1, c_1\}$ ,  $\{c_1, c_2\}$  and  $\{c_2, b_2\}$  is rooted.

Then the root curves of the three edges in the second condition are equal.

*Proof of Lemma 3.7 when  $p = 2$ .* Pick a 2-HBC  $a$  in  $S$  and choose a pentagon  $\Pi$  containing  $a$  and described as in Figure 3. Let  $(a, b_1, c_1, c_2, b_2)$  denote the 5-tuple defining  $\Pi$ . We note that  $b_1$  and  $b_2$  are 2-HBPs and that  $c_1$  and  $c_2$  are 1-HBPs. By Lemmas 3.6 and 3.8, the root curves of the three edges  $\{\phi(b_1), \phi(c_1)\}$ ,  $\{\phi(c_1), \phi(c_2)\}$  and  $\{\phi(c_2), \phi(b_2)\}$  are equal. Let  $\beta$  denote the common curve.

If  $\phi(a)$  were not a 2-HBC, then  $\phi(a)$  would be a 1-HBP and share a curve in  $S$  with each of  $\phi(b_1)$  and  $\phi(b_2)$ . We define  $b_1^2$  and  $b_2^2$  as the curves of  $\phi(b_1)$  and  $\phi(b_2)$  distinct from  $\beta$ , respectively. Since we have  $i(b_1^2, b_2^2) \neq 0$ ,  $\phi(a)$  has to contain  $\beta$ . This contradicts Lemma 3.1. It therefore turns out that  $\phi(a)$  is a 2-HBC, and the lemma follows.  $\square$

Before giving a proof of Lemma 3.7 when  $p \geq 3$ , let us prove the following:

**Lemma 3.9.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 3$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. For each non-separating curve  $\alpha$  in  $S$ , there exists a unique non-separating curve  $\beta$  in  $S$  such that for any HBP  $a$  in  $S$  containing  $\alpha$ , the HBP  $\phi(a)$  contains  $\beta$ .*

*Proof.* Pick a non-separating curve  $\alpha$  in  $S$ . Choosing two disjoint and distinct HBPs  $a, b$  in  $S$  containing  $\alpha$ , we define  $\beta$  to be the root curve of the rooted edge  $\{\phi(a), \phi(b)\}$ . The curve  $\beta$  depends only on  $\alpha$  thanks to Lemma 2.5 and the fact that for any two rooted edges  $\{a_1, b_1\}$  and  $\{a_2, b_2\}$  of  $\mathcal{CP}_n(S)$  whose root curves are equal to  $\alpha$ , there exists a sequence of rooted 2-simplices of  $\mathcal{CP}_n(S)$ ,  $\sigma_1, \dots, \sigma_m$ , such that  $a_1, b_1 \in \sigma_1$ ;  $a_2, b_2 \in \sigma_m$ ; and  $\sigma_j \cap \sigma_{j+1}$  is an edge of  $\mathcal{CP}_n(S)$  for each  $j = 1, \dots, m-1$ . Existence of such a sequence follows from Lemmas 4.5 and 5.1 in [22]. Uniqueness of  $\beta$  follows from injectivity of  $\phi$ .  $\square$

*Proof of Lemma 3.7 when  $p \geq 3$ .* Let  $\alpha$  be a  $j$ -HBC in  $S$ . We prove that  $\phi(\alpha)$  is a  $j$ -HBC in  $S$  by induction on  $j$ .

Let us assume  $j = 2$ . Assuming that  $\phi(\alpha)$  is an HBP, we deduce a contradiction. Pick a simplex  $s = \{\beta_1, \dots, \beta_p\}$  of  $\mathcal{C}(S)$  such that

- each curve of  $s$  is non-separating in  $S$  and is disjoint from  $\alpha$ ; and
- $\{\beta_1, \beta_2\}$  is a 2-HBP in  $S$ , and  $\{\beta_k, \beta_{k+1}\}$  is a 1-HBP in  $S$  for each  $k = 2, \dots, p-1$ .

By Lemma 2.6, there exists a simplex  $t = \{\gamma_1, \dots, \gamma_p\}$  of  $\mathcal{C}(S)$  with

$$\phi(\{\beta_k, \beta_l\}) = \{\gamma_k, \gamma_l\}$$

for any distinct  $k, l = 1, \dots, p$ . By Lemma 3.6,  $\{\gamma_1, \gamma_2\}$  is a 2-HBP in  $S$ , and  $\{\gamma_k, \gamma_{k+1}\}$  is a 1-HBP for each  $k = 2, \dots, p-1$ . We note that the HBP  $\phi(\alpha)$  is equivalent to  $\{\gamma_1, \gamma_p\}$  because the latter is a  $p$ -HBP and is disjoint from  $\phi(\alpha)$ . It follows from Proposition 2.3 that  $\phi(\alpha)$  contains at least one curve in  $t$ , say  $\gamma_{k_0}$ . Pick  $k_1 \in \{1, \dots, p\} \setminus \{k_0\}$  and choose a curve  $\delta$  in  $S$  such that

- $i(\delta, \beta_{k_0}) \neq 0$  and  $i(\delta, \alpha) = i(\delta, \beta_k) = 0$  for each  $k \in \{1, \dots, p\} \setminus \{k_0\}$ ; and
- $\{\beta_{k_1}, \delta\}$  is an HBP in  $S$ .

It then follows that  $\phi(\{\beta_{k_1}, \delta\})$  is disjoint from  $\phi(\alpha)$  and intersects  $\{\gamma_{k_1}, \gamma_{k_0}\}$ . Note that  $\phi(\{\beta_{k_1}, \delta\})$  contains  $\gamma_{k_1}$  by Lemma 3.9. The curve of  $\phi(\{\beta_{k_1}, \delta\})$  distinct from  $\gamma_{k_1}$  thus intersects  $\gamma_{k_0}$ . This is a contradiction because we have  $\gamma_{k_0} \in \phi(\alpha)$ .

We have proved that  $\phi(\alpha)$  is an HBC. Since  $\phi(\alpha)$  is disjoint from the 2-HBP  $\{\gamma_1, \gamma_2\}$  and the 1-HBP  $\{\gamma_k, \gamma_{k+1}\}$  for each  $k = 2, \dots, p-1$ , it has to be a 2-HBC.

We next assume  $j \geq 3$ . Pick a  $(j-1)$ -HBC  $\beta$  in  $S$  disjoint from  $\alpha$  and contained in the holed sphere cut off by  $\alpha$  from  $S$ . The hypothesis of the induction shows that  $\phi(\beta)$  is also a  $(j-1)$ -HBC. Let  $Q$  and  $R$  denote the components of  $S_\beta$  and  $S_{\phi(\beta)}$  of positive genus, respectively. The map  $\phi$  induces a superinjective map from  $\mathcal{CP}_n(Q)$  into  $\mathcal{CP}(R)$ . Since  $\alpha$  is a 2-HBC in  $Q$ ,  $\phi(\alpha)$  is a 2-HBC in  $R$  by Lemma 3.7 in the case  $p = 2$ . It thus turns out that  $\phi(\alpha)$  is a  $j$ -HBC in  $S$ .  $\square$

The following lemma is an immediate consequence of Lemmas 3.5 and 3.7.

**Lemma 3.10.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. Then the inclusion  $\phi(\mathcal{CP}_n(S)) \subset \mathcal{CP}_n(S)$  holds.*

#### 4. SUPERINJECTIVE MAPS FROM $\mathcal{CP}_s(S)$ INTO ITSELF

We discuss topological properties of vertices preserved by any superinjective map from  $\mathcal{CP}_s(S)$  into itself. Throughout this section, we mean by an HBP a separating one, unless otherwise stated.

We mean by a *hexagon* in  $\mathcal{CP}(S)$  a subgraph of  $\mathcal{CP}(S)$  consisting of six vertices  $v_1, \dots, v_6$  with  $i(v_k, v_{k+1}) = 0$ ,  $i(v_k, v_{k+2}) \neq 0$  and  $i(v_k, v_{k+3}) \neq 0$  for each  $k \bmod 6$ . In this case, let us say that the hexagon is defined by the 6-tuple  $(v_1, \dots, v_6)$ . A hexagon in  $\mathcal{CP}_s(S)$  (resp.  $\mathcal{CP}_n(S)$ ) is defined as a hexagon in  $\mathcal{CP}(S)$  each of whose vertices belongs to  $\mathcal{CP}_s(S)$  (resp.  $\mathcal{CP}_n(S)$ ). Examples of hexagons in  $\mathcal{CP}(S)$  are described in Figures 4, 5 and 6.

**Proposition 4.1.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Let  $(a, b, c, d, e, f)$  be a 6-tuple defining a hexagon in  $\mathcal{CP}_s(S)$  such that*

- *any of  $a, \dots, f$  is an HBP; and*
- *any of the four edges  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{d, e\}$  and  $\{e, f\}$  is rooted.*

*Then the following assertions hold:*

- The HBP  $a$  is equivalent to neither  $b$  nor  $f$ . In particular, we have  $g \geq 3$ .*
- Each of  $a, b, d$  and  $f$  is a 1-HBP, and each of  $c$  and  $e$  is a 2-HBP.*

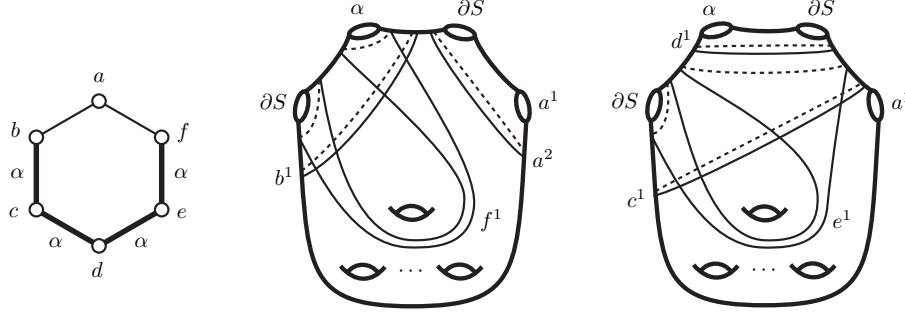


FIGURE 4. A hexagon in  $\mathcal{CP}_s(S_{g,2})$  with  $g \geq 3$  and  $a = \{a^1, a^2\}$ ,  $b = \{\alpha, b^1\}$ ,  $c = \{\alpha, c^1\}, \dots, f = \{\alpha, f^1\}$ .

Note that the hexagon in Figure 4 satisfies the assumption in this proposition. Let us summarize elementary properties on separating HBPs, which will be used to prove the proposition. One can readily check the following two lemmas.

**Lemma 4.2.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Let  $x$  and  $y$  be HBPs in  $S$  with  $\{x, y\}$  an edge of  $\mathcal{CP}_s(S)$ . Then the following two assertions hold:*

- (i) *If  $x$  and  $y$  are not equivalent, then  $x$  and  $y$  are 1-HBPs.*
- (ii) *If  $x$  and  $y$  are equivalent, then the edge  $\{x, y\}$  is rooted, and either  $x$  and  $y$  are 1-HBPs or one of  $x$  and  $y$  is a 1-HBP and another is a 2-HBP. The former is the case if and only if the root curve of  $\{x, y\}$  separates the two components of  $\partial S$ .*

**Lemma 4.3.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . The following assertions hold:*

- (i) *If  $x, y$  and  $z$  are 1-HBPs in  $S$  such that  $\{x, y\}$  and  $\{y, z\}$  are rooted edges of  $\mathcal{CP}_s(S)$ , then the root curves of  $\{x, y\}$  and  $\{y, z\}$  are equal.*
- (ii) *If  $y$  is a 1-HBP in  $S$  and  $x$  and  $z$  are 2-HBPs in  $S$  such that  $\{x, y\}$  and  $\{y, z\}$  are rooted edges of  $\mathcal{CP}_s(S)$ , then the root curves of  $\{x, y\}$  and  $\{y, z\}$  are equal.*
- (iii) *If  $x, y, z$  and  $w$  are mutually distinct 1-HBPs in  $S$  such that  $\{x, y\}$ ,  $\{y, z\}$  and  $\{z, w\}$  are rooted edges of  $\mathcal{CP}_s(S)$ , then we have  $i(w, x) = 0$  and the edge  $\{w, x\}$  is rooted.*

*Proof of Proposition 4.1 (i).* We denote by  $\Pi$  the hexagon in  $\mathcal{CP}_s(S)$  defined by the 6-tuple  $(a, b, c, d, e, f)$ . Assuming that the former assertion of assertion (i) is not true, we deduce a contradiction. Note that in this case,  $a$  is equivalent to both  $b$  and  $f$  because the six HBPs of  $\Pi$  associate the same curve in  $\tilde{S}$ , the closed surface obtained by attaching disks to all components of  $\partial S$ , under the inclusion of  $S$  into  $\tilde{S}$ . It then follows that the edges  $\{a, b\}$  and  $\{f, a\}$  are also rooted. We do not hence need to specify  $a$  and deduce a contradiction in the following four cases: (a)  $\Pi$  contains no 2-HBP; (b)  $\Pi$  contains exactly one 2-HBP; (c)  $\Pi$  contains exactly two 2-HBPs; and (d)  $\Pi$  contains three 2-HBPs.

Cases (a) and (b) are impossible by Lemma 4.3 (iii). In case (c), we first assume that  $a$  and  $d$  are 2-HBPs. Let  $Q_1$  denote the holed sphere cut off by  $a$  from  $S$ . The curves in  $b$  and  $f$  that do not belong to  $a$  intersect and thus fill  $Q_1$ . They belong to  $c$  and  $e$ , respectively, by Lemma 4.2 (ii). The equality  $a = d$  thus has to hold.

This is a contradiction. We next assume that  $c$  and  $e$  are 2-HBPs. By Lemma 4.3 (ii),  $c$ ,  $d$  and  $e$  share a curve, denoted by  $\alpha$ . Similarly, by Lemma 4.3 (i),  $b$ ,  $a$  and  $f$  share a curve, denoted by  $\beta$ . We have  $\alpha \neq \beta$  by Lemma 4.2 (ii). We define  $c^1$ ,  $d^1$  and  $e^1$  as the curves in  $c$ ,  $d$  and  $e$  distinct from  $\alpha$ , respectively. It follows from  $i(b, c) = i(d, c) = 0$  that  $\beta$  and  $d^1$  are curves in the holed sphere, denoted by  $Q_2$ , cut off by  $c$  from  $S$ . Since  $b$  consists of  $\beta$  and a curve in  $c$  and since we have  $i(b, d) \neq 0$ , the curves  $\beta$  and  $d^1$  intersect and fill  $Q_2$ . This contradicts the existence of the curve  $e^1$  satisfying  $i(e^1, \beta) = i(e^1, d) = 0$  and  $i(e^1, c^1) \neq 0$ .

In case (d), we may assume that  $a$ ,  $c$  and  $e$  are 2-HBPs. By Lemma 4.3 (ii), the three HBPs in each of the triplets  $\{c, d, e\}$ ,  $\{e, f, a\}$  and  $\{a, b, c\}$  share a curve in  $S$ , and we denote it by  $\alpha$ ,  $\beta$  and  $\gamma$ , respectively. Note that  $\alpha$ ,  $\beta$  and  $\gamma$  are mutually disjoint. If  $\alpha$ ,  $\beta$  and  $\gamma$  were mutually distinct, then we would have the equalities  $a = \{\beta, \gamma\}$ ,  $c = \{\gamma, \alpha\}$  and  $e = \{\alpha, \beta\}$ , and these three HBPs would form a 2-simplex of  $\mathcal{CP}_s(S)$ . This is a contradiction. If  $\alpha = \beta \neq \gamma$ , then we would have  $a = c$  and obtain a contradiction. A similar argument implies the equality  $\alpha = \beta = \gamma$ . We define  $a^1, \dots, f^1$  as the curves of  $a, \dots, f$  distinct from  $\alpha$ , respectively.

Let  $R$  denote the component of  $S_\alpha$  containing  $\partial S$ , which contains all the curves  $a^1, \dots, f^1$ . We have the three 3-HBCs  $a^1$ ,  $c^1$  and  $e^1$  in  $R$  encircling the two components of  $\partial S$  and the boundary component of  $R$ , denoted by  $\partial_\alpha$ , that corresponds to  $\alpha$ . The other three 2-HBCs  $b^1$ ,  $d^1$  and  $f^1$  in  $R$  encircle one component of  $\partial S$  and  $\partial_\alpha$ . Without loss of generality, we may assume that  $b^1$  and  $d^1$  encircle  $\partial_\alpha$  and the same component of  $\partial S$ , denoted by  $\partial_1$ . Another component of  $\partial S$  is denoted by  $\partial_2$ . Let  $\tilde{R}$  denote the surface obtained from  $R$  by attaching a disk to  $\partial_2$ . We have the simplicial map  $\rho: \mathcal{C}(R) \rightarrow \mathcal{C}^*(\tilde{R})$  associated with the inclusion of  $R$  into  $\tilde{R}$ , where  $\mathcal{C}^*(\tilde{R})$  is the simplicial cone over  $\mathcal{C}(\tilde{R})$  with the cone point  $*$ . Note that  $\rho^{-1}(*)$  consists of all 2-HBCs in  $R$  encircling  $\partial_2$  and another boundary component of  $R$ . We then obtain the equality

$$\rho(a^1) = \rho(b^1) = \rho(c^1) = \rho(d^1) = \rho(e^1).$$

If  $f^1$  encircles  $\partial_1$  and  $\partial_\alpha$ , then we have  $\rho(f^1) = \rho(a^1)$ . This contradicts Theorem 7.1 in [18] asserting that the inverse image of a curve in  $\tilde{R}$  under  $\rho$  is a simplicial tree. If  $f^1$  encircles  $\partial_2$  and  $\partial_\alpha$ , then the equality  $\rho(a^1) = \rho(e^1)$  implies the equality  $a^1 = e^1$  because  $a^1$  and  $e^1$  are disjoint from  $f^1$ . This is also a contradiction.

If  $g = 2$ , then any two disjoint and separating HBPs in  $S$  are equivalent. The existence of  $\Pi$  therefore implies  $g \geq 3$ .  $\square$

*Proof of Proposition 4.1 (ii).* Assertion (i) and Lemma 4.2 (i) imply that  $a$ ,  $b$  and  $f$  are 1-HBPs. By Lemma 4.3 (iii), at least one of  $c$ ,  $d$  and  $e$  is a 2-HBP. Assuming that exactly one of  $c$ ,  $d$  and  $e$  is a 2-HBP, we deduce a contradiction. This completes the proof.

First, we assume that  $c$  is a 2-HBP. By Lemma 4.3 (i),  $d$ ,  $e$  and  $f$  share a curve, denoted by  $\alpha$ . The curve  $\alpha$  separates the two components of  $\partial S$ . It follows that  $d^1$  and  $f^1$ , the curves of  $d$  and  $f$  distinct from  $\alpha$ , respectively, are contained in the same component of  $S_\alpha$  and that the two curves of  $a$  are contained in another component of  $S_\alpha$ . The equality  $i(a, d) = 0$  thus holds, and this is a contradiction. By symmetry, we can also deduce a contradiction if we assume that  $e$  is a 2-HBP.

We next assume that  $d$  is a 2-HBP. Let  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  denote the root curves of  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{d, e\}$  and  $\{e, f\}$ , respectively. By Lemma 4.2 (ii), we have  $\alpha \neq \beta$  and  $\gamma \neq \delta$ , and the equalities  $c = \{\alpha, \beta\}$  and  $e = \{\gamma, \delta\}$  thus hold. Since  $d$  contains

$\beta$  and  $\gamma$  and since we have  $i(c, e) \neq 0$ , the two curves  $\alpha$  and  $\delta$  intersect and fill the holed sphere cut off by  $d$  from  $S$ . This contradicts the existence of the 1-HBP  $a$  disjoint from  $\alpha$  and  $\delta$ .  $\square$

We now turn our attention to superinjective maps from  $\mathcal{CP}_s(S)$  into itself.

**Lemma 4.4.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then for each integer  $k$  with  $1 \leq k \leq p$ , any superinjective map from  $\mathcal{CP}_s(S)$  into itself preserves  $k$ -HBPs.*

We first prove this lemma when  $p = 2$ .

*Proof of Lemma 4.4 when  $p = 2$ .* If  $g \geq 3$ , then the lemma is obtained by using the hexagon in Figure 4 and Proposition 4.1. If  $g = 2$ , then pick a superinjective map  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  and the hexagon in  $\mathcal{CP}_s(S)$  described in Figure 6, which is defined by a 6-tuple  $(a, c_1, b_1, c_3, b_2, c_2)$  such that

- $a$  is a 2-HBC;  $b_1$  and  $b_2$  are 1-HBPs;  $c_1, c_2$  and  $c_3$  are 2-HBPs; and
- any of the four edges  $\{c_1, b_1\}$ ,  $\{b_1, c_3\}$ ,  $\{c_3, b_2\}$  and  $\{b_2, c_2\}$  is rooted.

By Proposition 4.1 (i),  $\psi(a)$  is not an HBP and is thus a 2-HBC. It follows that  $\psi$  preserves 2-HBCs. Since any HBP in  $S$  disjoint from a 2-HBC is a 2-HBP, the map  $\psi$  also preserves 2-HBPs and thus 1-HBPs.  $\square$

In the rest of the proof of Lemma 4.4, we assume  $p \geq 3$  and prove it in the cases of  $g = 2$  and  $g \geq 3$  separately.

*Proof of Lemma 4.4 when  $g = 2$  and  $p \geq 3$ .* Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. Note that any two disjoint and separating HBPs in  $S$  are equivalent thanks to the assumption  $g = 2$ . We claim that  $\psi$  preserves 2-HBCs.

Pick a 2-HBC  $\alpha$  in  $S$ . There then exists a simplex  $s = \{\beta_1, \dots, \beta_p\}$  of  $\mathcal{C}(S)$  disjoint from  $\alpha$  and consisting of mutually HBP-equivalent and separating curves in  $S$ . Let  $\sigma$  denote the simplex of  $\mathcal{CP}_s(S)$  consisting of all HBPs of two curves in  $s$ . If  $\psi(\alpha)$  were an HBP, then it would be equivalent to any HBP in  $\psi(\sigma)$ , and we can deduce a contradiction along the proof of Lemma 3.4. It thus turns out that  $\psi(\alpha)$  is an HBC. The existence of  $\sigma$  implies that  $\psi(\alpha)$  is a 2-HBC. Our claim follows.

We prove that  $\psi$  preserves  $k$ -HBPs by induction on  $p$ . The following argument appears in the proof of Lemma 3.15 in [22]. Pick a simplex  $\tau$  of  $\mathcal{CP}_s(S)$  of maximal dimension. Let  $\{\gamma_0, \dots, \gamma_p\}$  denote the collection of curves in HBPs of  $\tau$  so that  $\{\gamma_j, \gamma_{j+1}\}$  is a 1-HBP for each  $j = 0, \dots, p-1$ . By Lemma 2.6, there exist curves  $\delta_0, \dots, \delta_p$  in  $S$  with

$$\psi(\{\gamma_j, \gamma_k\}) = \{\delta_j, \delta_k\}$$

for any distinct  $j, k = 0, \dots, p$ . Choose two distinct 2-HBCs  $\epsilon_1$  and  $\epsilon_2$  in  $S$  contained in the holed sphere cut off by  $\{\gamma_0, \gamma_2\}$  from  $S$ . We now apply the hypothesis of the induction to the component of  $S_{\epsilon_1}$  of positive genus. It then follows that each of  $\{\delta_0, \delta_2\}$  and  $\{\delta_j, \delta_{j+1}\}$  for any  $j = 2, \dots, p-1$  is a 1-HBP in the component of  $S_{\psi(\epsilon_1)}$  of positive genus and that  $\{\delta_0, \delta_p\}$  is a  $(p-1)$ -HBP in that component.

Suppose that for some  $j = 2, \dots, p-1$ ,  $\{\delta_j, \delta_{j+1}\}$  is a 2-HBP in  $S$ . We choose a curve  $\gamma$  in  $S$  such that  $\{\gamma_0, \gamma\}$  is an HBP in  $S$ ; and  $\gamma$  intersects  $\gamma_{j+1}$  and is disjoint from  $\gamma_k$  for any  $k \in \{0, \dots, p\} \setminus \{j+1\}$ . Note that  $\{\gamma_0, \gamma\}$  is disjoint from  $\epsilon_1$  and  $\epsilon_2$ . On the other hand,  $\phi(\epsilon_1)$  and  $\phi(\epsilon_2)$  fill the holed sphere cut off by the 2-HBP  $\{\delta_j, \delta_{j+1}\}$  in  $S$ . Since the 2-simplex of  $\mathcal{CP}_s(S)$  consisting of  $\{\gamma_0, \gamma\}$ ,  $\{\gamma_0, \gamma_1\}$  and



$\{\gamma_0, \gamma_2\}$  is rooted, the HBP  $\phi(\{\gamma_0, \gamma\})$  contains  $\delta_0$ . Another curve of  $\phi(\{\gamma_0, \gamma\})$  intersects  $\delta_{j+1}$  and thus intersects  $\phi(\epsilon_1)$  or  $\phi(\epsilon_2)$ . This is a contradiction.

We have shown that  $\{\delta_0, \delta_2\}$  is a 2-HBP and that for each  $j = 2, \dots, p-1$ ,  $\{\delta_j, \delta_{j+1}\}$  is a 1-HBP. It follows that  $\{\delta_0, \delta_1\}$  and  $\{\delta_1, \delta_2\}$  are 1-HBPs and that  $\psi$  preserves  $k$ -HBPs for each  $k$ .  $\square$

*Proof of Lemma 4.4 when  $g \geq 3$  and  $p \geq 3$ .* Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. Pick a simplex  $\sigma$  of  $\mathcal{CP}_s(S)$  of maximal dimension. Let  $s = \{\beta_0, \dots, \beta_p\}$  denote the collection of curves in HBPs of  $\sigma$  with  $\{\beta_j, \beta_{j+1}\}$  a 1-HBP for each  $j = 0, \dots, p-1$ . Since we have  $g \geq 3$ , we may assume that the component, denoted by  $Q$ , of  $S_\sigma$  of positive genus containing  $\beta_p$  as a boundary component is of genus at least two. By Lemma 2.6, there exist curves  $\gamma_0, \dots, \gamma_p$  in  $S$  with

$$\psi(\{\beta_j, \beta_k\}) = \{\gamma_j, \gamma_k\}$$

for any distinct  $j, k = 0, \dots, p$ . Since the genus of  $Q$  is at least two, there exists a simplex  $\tau$  of  $\mathcal{CP}_s(S)$  consisting of  $p(p-1)/2$  HBPs which are mutually equivalent and any of which is disjoint from  $\{\beta_0, \beta_1\}$ , but not equivalent to it. It follows from Lemma 3.4 (ii) that  $\{\gamma_0, \gamma_1\}$  and each HBP in  $\psi(\tau)$  are not equivalent. We thus see that  $\{\gamma_0, \gamma_1\}$  is a 1-HBP and that there is a component of  $S_{\{\gamma_0, \gamma_1\}}$  of positive genus which has exactly one boundary component. Such a component of  $S_{\{\gamma_0, \gamma_1\}}$  uniquely exists and is denoted by  $R$ .

Choose a curve  $\beta$  in  $S$  such that  $\{\beta_0, \beta\}$  is an HBP and we have  $i(\beta, \beta_p) = 0$  and  $i(\beta, \beta_j) \neq 0$  for any  $j = 1, \dots, p-1$ . Since  $\{\beta_0, \beta\}$  and  $\{\beta_0, \beta_p\}$  form a rooted edge of  $\mathcal{CP}_s(S)$ , Lemma 2.5 shows that  $\psi(\{\beta_0, \beta\})$  contains  $\gamma_0$  or  $\gamma_p$ . Another curve, denoted by  $\gamma$ , of  $\psi(\{\beta_0, \beta\})$  intersects  $\gamma_j$  for any  $j = 1, \dots, p-1$ . It then follows that  $R$  contains  $\gamma_0$  as a boundary component because otherwise  $R$  would contain  $\gamma_1$  as a boundary component, and  $\gamma$  could not intersect  $\gamma_1$  and  $\gamma_2$  simultaneously, kept disjoint from  $\gamma_0$ .

It is shown that  $\{\gamma_0, \gamma_2\}$  is a 2-HBP by choosing a simplex of  $\mathcal{CP}_s(S)$  consisting of  $(p-1)(p-2)/2$  HBPs which are mutually equivalent and any of which is disjoint from  $\{\beta_0, \beta_2\}$ , but not equivalent to it. Repeating this argument, we see that for each  $j = 1, \dots, p$ ,  $\{\gamma_0, \gamma_j\}$  is a  $j$ -HBP.  $\square$

**Lemma 4.5.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then for each integer  $j$  with  $2 \leq j \leq p$ , any superinjective map from  $\mathcal{CP}_s(S)$  into itself preserves  $j$ -HBCs.*

*Proof of Lemma 4.5 when  $p = 2$ .* Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. It follows from Lemma 4.4 that the image of the hexagon  $\Pi$  in Figure 6 under  $\psi$  contains three 2-HBPs and four rooted edges. Proposition 4.1 implies that the image of the 2-HBC in  $\Pi$  have to be a 2-HBC. This proves that  $\psi$  preserves 2-HBCs.  $\square$

The proof of Lemma 4.5 when  $p \geq 3$  is a verbatim translation of that of Lemma 3.7 when  $p \geq 3$ . In place of Lemma 3.9, we use the following lemma, which can be verified by using Lemmas 4.10 and 5.3 in [22].

**Lemma 4.6.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 3$ , and let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. For each separating curve  $\alpha$  in  $S$ , there exists a unique separating curve  $\beta$  in  $S$  such that for any HBP  $a$  in  $S$  containing  $\alpha$ , the HBP  $\psi(a)$  contains  $\beta$ .*

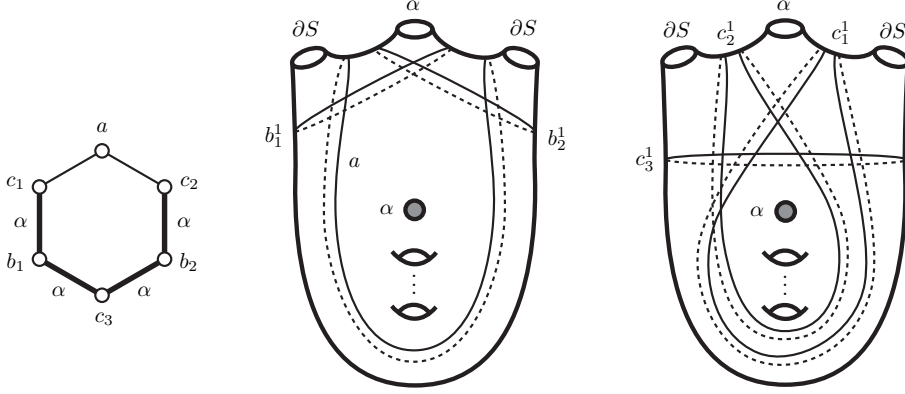


FIGURE 5. A hexagon in  $\mathcal{CP}_n(S_{g,2})$  with  $b_j = \{\alpha, b_j^1\}$  and  $c_k = \{\alpha, c_k^1\}$  for each  $j = 1, 2$  and each  $k = 1, 2, 3$ .

## 5. HEXAGONS IN $\mathcal{CP}_n(S_{g,2})$

In this section, we focus on the case  $p = 2$  and study topological types of hexagons in  $\mathcal{CP}_n(S_{g,2})$ . We then show that any superinjective map  $\phi$  from  $\mathcal{CP}_n(S_{g,2})$  into itself sends the link of any 2-HBP  $a$  onto the link of  $\phi(a)$ . Results in this section will also be used in the subsequent sections. Throughout this section, we put  $S = S_{g,2}$  with  $g \geq 2$  and mean by an HBP a non-separating one, unless otherwise stated, since we mainly deal with  $\mathcal{CP}_n(S)$ .

**Proposition 5.1.** *Let  $(a, c_1, b_1, c_3, b_2, c_2)$  be a 6-tuple defining a hexagon in  $\mathcal{CP}_n(S)$  such that*

- *$a$  is a 2-HBC;  $b_1$  and  $b_2$  are 1-HBPs;  $c_1, c_2$  and  $c_3$  are 2-HBPs; and*
- *any of the four edges  $\{c_1, b_1\}$ ,  $\{b_1, c_3\}$ ,  $\{c_3, b_2\}$  and  $\{b_2, c_2\}$  is rooted.*

*Then the following two assertions hold:*

- The root curves of the four edges in the second condition are equal.*
- We have  $i(b_1, b_2) = i(b_2, a) = i(a, b_1) = 2$ .*

The hexagon in Figure 5 satisfies the assumption in this proposition.

*Proof of Proposition 5.1 (i).* Let  $\alpha, \beta, \gamma$  and  $\delta$  denote the root curves of the edges  $\{c_1, b_1\}$ ,  $\{b_1, c_3\}$ ,  $\{c_3, b_2\}$  and  $\{b_2, c_2\}$ , respectively. First, assuming  $\beta \neq \gamma$ , we deduce a contradiction. We then have  $c_3 = \{\beta, \gamma\}$  and denote by  $Q$  the holed sphere cut off by  $c_3$  from  $S$ . We define a curve  $c_2^1$  so that  $c_2 = \{\delta, c_2^1\}$ . We assume that any two of these curves intersect minimally unless they are isotopic.

Suppose that the equality  $\alpha = \beta$  holds. If the equality  $\gamma = \delta$  were true, then we would have  $i(a, c_3) = 0$  and a contradiction. We thus have  $\gamma \neq \delta$  and  $b_2 = \{\gamma, \delta\}$ . Note that  $\delta$  is a curve in  $Q$ . Since  $c_2^1$  intersects  $\beta$  and is disjoint from  $\delta$  and  $\gamma$ , the intersection  $c_2^1 \cap Q$  consists of mutually isotopic, essential simple arcs in  $Q$  each of which cuts off an annulus containing exactly one component of  $\partial S$ . It is then impossible to find a position of  $a$  because  $a$  is a 2-HBC disjoint from  $\beta = \alpha$  and  $c_2$ .

We thus proved  $\alpha \neq \beta$ . Similarly, we have  $\gamma \neq \delta$  and the equalities  $b_1 = \{\alpha, \beta\}$ ,  $c_3 = \{\beta, \gamma\}$  and  $b_2 = \{\gamma, \delta\}$ . It turns out that  $\alpha$  and  $\delta$  intersect in  $Q$  and thus fill  $Q$ . Since  $a$  is disjoint from  $\alpha$  and  $\delta$ , it is impossible that  $a$  is a 2-HBC.

We have shown the equality  $\beta = \gamma$ . In what follows, we deduce a contradiction in the following three cases: (a)  $\alpha, \beta$  and  $\delta$  are mutually distinct; (b) either  $\alpha = \beta \neq \delta$  or  $\alpha \neq \beta = \delta$ ; and (c)  $\alpha = \delta \neq \beta$ .

In case (a), we have  $b_1 = \{\alpha, \beta\}$  and  $b_2 = \{\beta, \delta\}$ . It then follows that  $\alpha$  and  $\delta$  fill the holed sphere cut off by  $c_3$  from  $S$ . Since  $a$  is disjoint from  $\alpha$  and  $\delta$ , it is impossible that  $a$  is a 2-HBC. In case (b), we have either  $b_2 = \{\alpha, \delta\}$  or  $b_1 = \{\alpha, \delta\}$ . Each of the two equalities contradicts the condition that  $a$  is disjoint from  $\alpha$  and  $\delta$ , but intersects  $b_1$  and  $b_2$ . In case (c), we have the equality  $b_1 = b_2 = \{\alpha, \beta\}$  and a contradiction.  $\square$

*Proof of Proposition 5.1 (ii).* Let  $\alpha$  denote the root curve of the edges  $\{c_1, b_1\}$ ,  $\{b_1, c_3\}$ ,  $\{c_3, b_2\}$  and  $\{b_2, c_2\}$ . We define curves  $b_j^1$  and  $c_k^1$  so that  $b_j = \{\alpha, b_j^1\}$  and  $c_k = \{\alpha, c_k^1\}$  for each  $j = 1, 2$  and each  $k = 1, 2, 3$ . We assume that any two of these curves intersect minimally.

Let  $Q$  denote the holed sphere cut off by  $c_3$  from  $S$ . Note that  $b_1^1$  and  $b_2^1$  are curves in  $Q$ . For each  $j = 1, 2$ , using  $b_j^1$ , we see that the intersection  $c_j^1 \cap Q$  consists of mutually isotopic, essential simple arcs in  $Q$  each of which cuts off an annulus containing exactly one component of  $\partial S$ , denoted by  $\partial_j$ , and joins two points of  $c_3^1$ . Let  $l_j$  denote a component of  $c_j^1 \cap Q$ . It follows from  $b_1^1 \neq b_2^1$  that  $l_1$  and  $l_2$  are not isotopic. If  $l_1$  and  $l_2$  could not be isotoped so that they are disjoint, then the union of a subarc of  $l_1$  and a subarc of  $l_2$  would be a simple closed curve in  $Q$  isotopic to  $\partial_1$ . This is a contradiction because  $a$  is a 2-HBC in  $S$  disjoint from  $c_1$  and  $c_2$ . It thus turns out that  $l_1$  and  $l_2$  are non-isotopic and can be isotoped so that they are disjoint. The equality  $i(b_1, b_2) = i(b_1^1, b_2^1) = 2$  now follows.

Replacing the 6-tuple  $(a, c_1, b_1, c_3, b_2, c_2)$  with the 6-tuple  $(b_1, c_3, b_2, c_2, a, c_1)$ , we show  $i(b_2, a) = 2$ . Let  $R$  denote the holed sphere cut off by  $c_2^1$  from  $S_\alpha$ . Let  $\partial$  denote the boundary component of  $S_\alpha$  that corresponds to  $\alpha$  and is contained in  $R$ . The holed sphere cut off by  $b_2^1$  from  $S_\alpha$  contains  $\partial$  because  $b_2^1$  is disjoint from  $c_2^1$ . Repeating this argument, we see that the holed sphere cut off by each of  $c_3^1, b_1^1$  and  $c_1^1$  from  $S_\alpha$  contains  $\partial$ . As in the last paragraph, it is shown that the intersection  $c_1^1 \cap R$  consists of mutually isotopic, essential simple arcs in  $R$  each of which cuts off an annulus containing  $\partial$  and joins two points of  $c_2^1$ . The intersection  $c_3^1 \cap R$  also consists of mutually isotopic, essential simple arcs in  $R$ . Let  $r_1$  and  $r_3$  be components of  $c_1^1 \cap R$  and  $c_3^1 \cap R$ , respectively. It follows from  $a \neq b_2^1$  that  $r_1$  and  $r_3$  are not isotopic. If  $r_1$  and  $r_3$  could not be isotoped so that they are disjoint, then the union of a subarc of  $r_1$  and a subarc of  $r_3$  would be a simple closed curve in  $R$  isotopic to  $\partial$ . This is a contradiction because  $b_1^1$  is a 2-HBC in  $S_\alpha$  which is disjoint from  $c_1^1$  and  $c_3^1$  and cuts off a pair of pants containing  $\partial$  from  $S_\alpha$ . It follows that  $r_1$  and  $r_3$  are non-isotopic and can be isotoped so that they are disjoint. The equality  $i(b_2, a) = i(b_2^1, a) = 2$  is obtained.

Along a verbatim argument, we obtain the equality  $i(a, b_1) = 2$ .  $\square$

As an application of the last proposition, surjectivity of a superinjective map on the link of a 2-HBP is verified.

**Lemma 5.2.** *Let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  be a superinjective map. Then for each 2-HBP  $b$  in  $S$ , we have the equality*

$$\phi(\text{Lk}(b)) = \text{Lk}(\phi(b)),$$

where for each vertex  $c$  of  $\mathcal{CP}_n(S)$ ,  $\text{Lk}(c)$  denotes the link of  $c$  in  $\mathcal{CP}_n(S)$ .

Before proving this lemma, we recall the simplicial graph associated to  $S_{0,4}$ .

**Graph  $\mathcal{F}(X)$ .** For a surface  $X$  homeomorphic to  $S_{0,4}$ , we define a simplicial graph  $\mathcal{F}(X)$  so that the set of vertices of  $\mathcal{F}(X)$  is  $V(X)$  and two elements  $\alpha, \beta \in V(X)$  are connected by an edge of  $\mathcal{F}(X)$  if and only if we have  $i(\alpha, \beta) = 2$ .

This graph is known to be isomorphic to the Farey graph (see Section 3.2 in [24]). It is shown that any injective simplicial map from the Farey graph into itself is surjective. To prove Lemma 5.2, we also need the following:

**Lemma 5.3.** *Let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  be a superinjective map. Suppose that for each  $k = 1, 2, 3, 4$ , we have a non-separating HBP  $a_k = \{\alpha, \alpha_k\}$  in  $S$  such that  $\{a_1, a_2\}$  and  $\{a_3, a_4\}$  are edges of  $\mathcal{CP}_n(S)$ . Then the root curves of the two edges  $\{\phi(a_1), \phi(a_2)\}$  and  $\{\phi(a_3), \phi(a_4)\}$  of  $\mathcal{CP}_n(S)$  are equal.*

*Proof.* Since we have already shown that  $\phi$  preserves 2-HBCs, 1-HBPs and 2-HBPs, respectively, the proof of Lemma 4.7 in [22] is now valid for the present setting.  $\square$

*Proof of Lemma 5.2.* We define curves  $\beta_1, \beta_2, \gamma_1$  and  $\gamma_2$  so that the equalities

$$b = \{\beta_1, \beta_2\}, \quad \phi(b) = \{\gamma_1, \gamma_2\}$$

hold. By Lemma 5.3, we may assume that for any  $j = 1, 2$  and any rooted edge  $e$  of  $\mathcal{CP}_n(S)$  whose root curve is equal to  $\beta_j$ , the root curve of  $\phi(e)$  is equal to  $\gamma_j$ . It follows from Lemmas 3.6 and 3.7 that  $\phi$  induces a map from the union of the set of HBCs in  $\text{Lk}(b)$  and the set of HBPs in  $\text{Lk}(b)$  containing  $\beta_1$  into the union of the set of HBCs in  $\text{Lk}(\phi(b))$  and the set of HBPs in  $\text{Lk}(\phi(b))$  containing  $\gamma_1$ . Applying Proposition 5.1 (ii), we conclude that this map induces an injective simplicial map between the Farey graphs associated to the holed spheres cut off by  $b$  and by  $\phi(b)$  from  $S$ . Since such a simplicial map is surjective, the set  $\phi(\text{Lk}(b))$  contains all HBCs in  $\text{Lk}(\phi(b))$  and all HBPs in  $\text{Lk}(\phi(b))$  containing  $\gamma_1$ . In a similar way, we can show that  $\phi(\text{Lk}(b))$  contains all HBPs in  $\text{Lk}(\phi(b))$  containing  $\gamma_2$ .  $\square$

Let us denote by  $\mathcal{H}_n = \mathcal{H}_n(S)$  the set of all hexagons in  $\mathcal{CP}_n(S)$  satisfying the assumption in Proposition 5.1. The following lemma motivates us to relate the link of a 1-HBP  $b$  in  $\mathcal{CP}_n(S)$  with the complex of arcs for the component of  $S_b$  of positive genus. The latter complex will be examined in Section 7.

**Lemma 5.4.** *Let  $b_1$  be a 1-HBP in  $S$  and pick a curve  $\alpha$  in  $b_1$ . We denote by  $X$  the component of  $S_{b_1}$  of positive genus. Let  $c_1 = \{\alpha, c_1^1\}$  and  $c_3 = \{\alpha, c_3^1\}$  be 2-HBPs in  $S$  with  $c_1 \neq c_3$  and  $i(c_1, b_1) = i(c_3, b_1) = 0$ . Then there exists a hexagon in  $\mathcal{H}_n$  containing  $b_1, c_1$  and  $c_3$  if and only if the defining arcs of  $c_1^1$  and  $c_3^1$  as curves in  $X$  can be disjoint.*

*Proof.* Suppose that there exists a hexagon  $\Pi$  in  $\mathcal{H}_n$  containing  $b_1, c_1$  and  $c_3$ . We then have a 6-tuple  $(a, c_1, b_1, c_3, b_2, c_2)$  defining  $\Pi$ . For each  $j = 1, 2$ , let  $b_j^1$  denote the curve of  $b_j$  distinct from  $\alpha$ . Let  $l_a, l_1$  and  $l_2$  denote the defining arcs of  $a, b_1^1$  and  $b_2^1$  as curves in  $S_\alpha$ . Since  $b_1^1$  and  $b_2^1$  are curves in the holed sphere cut off by  $c_3$  from  $S$ , the arcs  $l_1$  and  $l_2$  meet the same boundary component of  $S_\alpha$  corresponding to  $\alpha$ . Proposition 5.1 (ii) implies that  $l_a, l_1$  and  $l_2$  can mutually be disjoint. The defining arcs of  $c_1^1$  and  $c_3^1$  as curves in  $X$  are  $l_a \cap X$  and  $l_2 \cap X$ , respectively, and are thus disjoint.

Conversely, suppose that the defining arcs of  $c_1^1$  and  $c_3^1$  as curves in  $X$ , denoted by  $r_1$  and  $r_3$ , respectively, are disjoint. Label as  $\partial_1$  and  $\partial_2$  the components of  $\partial S$

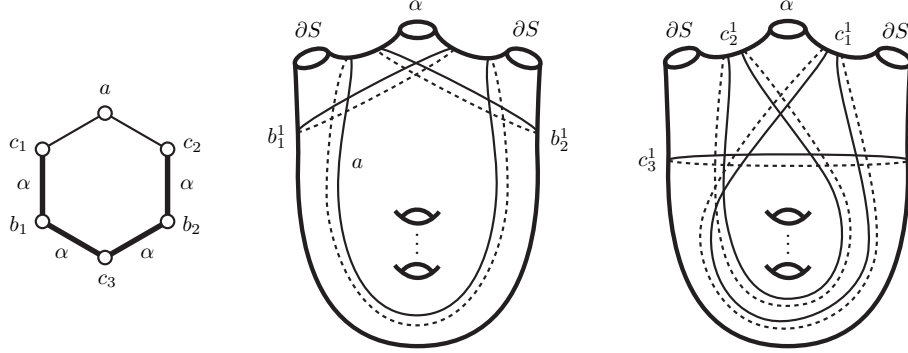


FIGURE 6. A hexagon in  $\mathcal{CP}_s(S_{g,2})$  with  $b_j = \{\alpha, b_j^1\}$  and  $c_k = \{\alpha, c_k^1\}$  for each  $j = 1, 2$  and each  $k = 1, 2, 3$ .

so that  $\partial_2$  is contained in the pair of pants cut off by  $b_1$  from  $S$ . Let  $\partial_3$  denote the boundary component of  $S_\alpha$  that corresponds to  $\alpha$  and is contained in the holed sphere cut off by  $c_1^1$  from  $S_\alpha$ . The component  $\partial_3$  is also contained in the holed sphere cut off by  $c_3^1$  from  $S_\alpha$ . Each of  $r_1$  and  $r_3$  connects the two component of  $\partial X$  corresponding to  $\partial_1$  and  $b_1^1$ , the curve of  $b_1$  distinct from  $\alpha$ .

We now extend  $r_1$  to an essential simple arc  $l_a$  in  $S_\alpha$  which connects  $\partial_1$  and  $\partial_2$ . We next extend  $r_3$  to an essential simple arc  $l_2$  in  $S_\alpha$  which connects  $\partial_1$  and  $\partial_3$  and is disjoint from  $l_a$ . Let  $a$  and  $b_2^1$  denote the curves in  $S_\alpha$  defined by  $l_a$  and  $l_2$ , respectively. Since  $l_a$  and  $l_2$  are disjoint, there exists a curve  $c_2^1$  disjoint from  $a$  and  $b_2^1$  and with the pair  $\{\alpha, c_2^1\}$  a 2-HBP in  $S$ . Such a curve is unique up to isotopy.

We check that the 6-tuple  $(a, c_1, b_1, c_3, b_2, c_2)$  defines a hexagon in  $\mathcal{H}_n$ , where  $b_2 = \{\alpha, b_2^1\}$  and  $c_2 = \{\alpha, c_2^1\}$ . Let  $l_1$  denote the defining arc of  $b_1^1$  as a curve in  $S_\alpha$ , which connects  $\partial_2$  and  $\partial_3$ . Since  $l_2$  connects  $\partial_1$  and  $\partial_3$ , we have  $i(b_1^1, b_2^1) \neq 0$ . Similarly, we have  $i(b_1^1, a) \neq 0$  and  $i(a, b_2^1) \neq 0$ . If  $c_2^1$  were disjoint from  $b_1^1$ , then we would have  $c_2^1 = c_1^1$  because  $c_1^1$  is a boundary component of a regular neighborhood of  $b_1^1 \cup a$  in  $S_\alpha$ . We also have  $c_2^1 = c_3^1$  because of the same reason. This contradicts  $c_1^1 \neq c_3^1$ . We thus have  $i(b_1^1, c_2^1) \neq 0$ . In particular,  $c_1^1$ ,  $c_2^1$  and  $c_3^1$  are mutually distinct. The same kind of argument shows that  $i(c_3^1, a) \neq 0$  and  $i(c_1^1, b_2^1) \neq 0$ . Since  $c_1$ ,  $c_2$  and  $c_3$  are mutually distinct 2-HBPs in  $S$  containing  $\alpha$ , the curves  $c_1^1$ ,  $c_2^1$  and  $c_3^1$  mutually intersect.  $\square$

## 6. HEXAGONS IN $\mathcal{CP}_s(S_{g,2})$

In this section, we study hexagons in  $\mathcal{CP}_s(S_{g,2})$  and obtain results similar to those in the previous section (see Figure 6 for such a hexagon). The proof is also similar, and its large part is thus omitted. Throughout this section, we put  $S = S_{g,2}$  with  $g \geq 2$  and mean by an HBP a separating one, unless otherwise stated, since we mainly deal with  $\mathcal{CP}_s(S)$ .

**Proposition 6.1.** *Let  $(a, c_1, b_1, c_3, b_2, c_2)$  be a 6-tuple defining a hexagon in  $\mathcal{CP}_s(S)$  such that*

- *$a$  is a 2-HBC;  $b_1$  and  $b_2$  are 1-HBPs;  $c_1$ ,  $c_2$  and  $c_3$  are 2-HBPs; and*
- *any of the four edges  $\{c_1, b_1\}$ ,  $\{b_1, c_3\}$ ,  $\{c_3, b_2\}$  and  $\{b_2, c_2\}$  is rooted.*

*Then the following two assertions hold:*

- (i) *The root curves of the four edges in the second condition are equal.*
- (ii) *We have  $i(b_1, b_2) = i(b_2, a) = i(a, b_1) = 2$ .*

*Proof.* Assertion (i) is proved in Lemma 4.12 of [22]. Let  $\alpha$  denote the common curve in assertion (i). Let  $R$  denote the component of  $S_\alpha$  containing  $\partial S$ , whose genus is positive and less than  $g$ . For each  $j = 1, 2$  and each  $k = 1, 2, 3$ , we define curves  $b_j^1$  and  $c_k^1$  so that  $b_j = \{\alpha, b_j^1\}$  and  $c_k = \{\alpha, c_k^1\}$ . Let  $Q$  denote the holed sphere cut off by  $c_3^1$  from  $R$ . It follows that  $a$  and each  $c_k^1$  are curves in  $R$  and that each  $b_j^1$  is a curve in  $Q$ . Along an argument of the same kind as in the proof of Proposition 5.1 (ii), we obtain the equality in assertion (ii).  $\square$

**Lemma 6.2.** *Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. Then for each 2-HBP  $b$  in  $S$ , we have the equality*

$$\psi(\text{Lk}(b)) = \text{Lk}(\psi(b)),$$

*where for each vertex  $c$  of  $\mathcal{CP}_s(S)$ ,  $\text{Lk}(c)$  denotes the link of  $c$  in  $\mathcal{CP}_s(S)$ .*

The proof of this lemma is a verbatim translation of the proof of Lemma 5.2 once the following lemma is obtained.

**Lemma 6.3.** *Let  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be a superinjective map. Suppose that for each  $k = 1, 2, 3, 4$ , we have an HBP  $a_k = \{\alpha, \alpha_k\}$  in  $S$  such that  $\{a_1, a_2\}$  and  $\{a_3, a_4\}$  are edges of  $\mathcal{CP}_s(S)$ . Then the root curves of the two edges  $\{\psi(a_1), \psi(a_2)\}$  and  $\{\psi(a_3), \psi(a_4)\}$  of  $\mathcal{CP}_s(S)$  are equal.*

*Proof.* Since we have already shown that  $\psi$  preserves 2-HBCs, 1-HBPs and 2-HBPs, respectively, the proof of Lemma 4.14 in [22] is now valid for the present setting.  $\square$

Let us denote by  $\mathcal{H}_s = \mathcal{H}_s(S)$  the set of all hexagons in  $\mathcal{CP}_s(S)$  satisfying the assumption in Proposition 6.1. Along an argument of the same kind as in the proof of Lemma 5.4, we can show the following:

**Lemma 6.4.** *Let  $b_1$  be a 1-HBP in  $S$  and let  $\alpha$  denote the curve in  $b_1$  that does not separate the two components of  $\partial S$ . We denote by  $Y$  the component of  $S_{b_1}$  of positive genus and containing the curve of  $b_1$  distinct from  $\alpha$  as a boundary component. Let  $c_1 = \{\alpha, c_1^1\}$  and  $c_3 = \{\alpha, c_3^1\}$  be 2-HBPs in  $S$  with  $c_1 \neq c_3$  and  $i(c_1, b_1) = i(c_3, b_1) = 0$ . Then there exists a hexagon in  $\mathcal{H}_s$  containing  $b_1$ ,  $c_1$  and  $c_3$  if and only if the defining arcs of  $c_1^1$  and  $c_3^1$  as curves in  $Y$  can be disjoint.*

## 7. THE COMPLEXES $\mathcal{D}(X, \partial)$ AND $\mathcal{D}(Y)$

We introduce natural subcomplexes of the complexes of arcs for certain surfaces, motivated by Lemmas 5.4 and 6.4 that describe a relationship between the link of a 1-HBP  $a$  in  $\mathcal{CP}_n(S)$  or  $\mathcal{CP}_s(S)$  and the complex of arcs for a component of  $S_a$  of positive genus. Let us recall the complex of arcs for a surface.

**Complex  $\mathcal{A}(S)$ .** Let  $S$  be a surface with non-empty boundary. We define  $V_a(S)$  as the set of isotopy classes of essential simple arcs in  $S$ . Let  $\mathcal{A}(S)$  denote the abstract simplicial complex whose  $n$ -simplices are defined as a subset  $\sigma$  of  $V_a(S)$  such that we have  $|\sigma| = n + 1$  and have mutually disjoint representatives of elements of  $\sigma$ . We often identify an element of  $V_a(S)$  with its representative if there is no confusion.

**Complexes  $\mathcal{D}(X, \partial)$  and  $\mathcal{D}(Y)$ .** Let  $X = S_{g,3}$  be a surface with  $g \geq 1$  and fix a boundary component  $\partial$  of  $X$ . We define  $\mathcal{D}(X, \partial)$  as the full subcomplex of  $\mathcal{A}(X)$

spanned by all vertices that correspond to arcs in  $X$  connecting a component of  $\partial X \setminus \partial$  with another component of  $\partial X \setminus \partial$ .

Let  $Y = S_{g,2}$  be a surface with  $g \geq 1$ . We define  $\mathcal{D}(Y)$  as the full subcomplex of  $\mathcal{A}(Y)$  spanned by all vertices that correspond to arcs in  $Y$  connecting a component of  $\partial Y$  with another component of  $\partial Y$ .

The aim of this section is to show the following:

**Proposition 7.1.** *In the above notation, any injective simplicial map from  $\mathcal{D}(X, \partial)$  into itself is surjective. Moreover, any injective simplicial map from  $\mathcal{D}(Y)$  into itself is also surjective.*

The first subsection is preliminary to the proof of this proposition. We introduce variants of the complex of arcs and show that most of them are connected. These variants are special ones of the complex introduced and denoted by  $BZ(\Delta, \Delta^0)$  in [9]. The proof of Proposition 7.1 is presented in the second subsection.

**7.1. Connectivity of complexes of arcs.** Throughout this subsection, we fix a surface  $Q = S_{g,p+1}$  with  $g \geq 0$  and  $p \geq 1$ , and fix a boundary component  $\partial_0$  of  $Q$ . Let  $R$  be the surface obtained from  $Q$  by attaching a disk to  $\partial_0$ . We label the components of  $\partial Q$  other than  $\partial_0$  as  $\partial_1, \dots, \partial_p$ . Suppose that we have a subset  $\Delta$  of  $\partial Q \setminus \partial_0$  with its decomposition  $\Delta = \Delta^+ \sqcup \Delta^-$  satisfying the following two conditions: For each  $j = 1, \dots, p$ ,

- the two sets  $\Delta^+ \cap \partial_j$  and  $\Delta^- \cap \partial_j$  are non-empty, finite and have the same cardinality; and
- along  $\partial_j$ , points of  $\Delta^+ \cap \partial_j$  and  $\Delta^- \cap \partial_j$  appear alternatively.

**Complexes  $\mathcal{A}(Q, \Delta)$  and  $\mathcal{A}(R, \Delta)$ .** We denote by  $V_a(Q, \Delta)$  the set of isotopy classes relative to  $\Delta$  of simple arcs  $l$  in  $Q$  such that

- $l$  connects a point of  $\Delta^+$  with a point of  $\Delta^-$  and meets  $\partial Q$  only at its end points; and
- $l$  is not homotopic relative to  $\Delta$  to an arc in  $\partial Q$  which meets  $\Delta$  only at its end points.

Let  $\mathcal{A}(Q, \Delta)$  denote the abstract simplicial complex whose  $n$ -simplices are defined as a subset  $\sigma$  of  $V_a(Q, \Delta)$  such that we have  $|\sigma| = n + 1$  and have representatives  $l_0, \dots, l_n$  of elements of  $\sigma$  with  $l_j \cap l_k = \partial l_j \cap \partial l_k$  for any distinct  $j$  and  $k$ .

The set  $V_a(R, \Delta)$  and the complex  $\mathcal{A}(R, \Delta)$  are defined in the same manner after replacing  $Q$  with  $R$  in the last paragraph. We often identify an element of  $V_a(Q, \Delta)$  or  $V_a(R, \Delta)$  with its representative if there is no confusion.

In the rest of this subsection, we prove the following:

**Proposition 7.2.** *We put  $n = |\Delta^+ \cap \partial_1| = |\Delta^- \cap \partial_1|$ . Then we have the following two assertions:*

- If  $\mathcal{A}(Q, \Delta)$  is not connected, then we have  $(g, p) = (0, 1)$  and  $n \leq 2$ . If  $(g, p, n) = (0, 1, 2)$ , then  $\mathcal{A}(Q, \Delta)$  is of dimension zero and consists of four vertices. If  $(g, p, n) = (0, 1, 1)$ , then  $\mathcal{A}(Q, \Delta) = \emptyset$ .*
- If  $\mathcal{A}(R, \Delta)$  is not connected, then we have  $(g, p) = (0, 1)$  and  $n \leq 3$ . If  $(g, p, n) = (0, 1, 3)$ , then  $\mathcal{A}(R, \Delta)$  is of dimension zero and consists of three vertices. If  $(g, p, n) = (0, 1, 2), (0, 1, 1)$ , then  $\mathcal{A}(R, \Delta) = \emptyset$ .*

As already mentioned, the complexes  $\mathcal{A}(Q, \Delta)$  and  $\mathcal{A}(R, \Delta)$  are special ones of the complex introduced and denoted by  $BZ(\Delta, \Delta^0)$  in [9], where high connectivity of it is discussed. The proof however does not care connectivity of it when  $g$  and  $p$  are small. We hence present a direct proof of Proposition 7.2.

In the case of  $(g, p) = (0, 1)$ , both  $\mathcal{A}(Q, \Delta)$  and  $\mathcal{A}(R, \Delta)$  consist of finitely many vertices. The assertion in Proposition 7.2 for this case can directly be checked. In what follows, we assume  $(g, p) \neq (0, 1)$  and prove that  $\mathcal{A}(R, \Delta)$  is connected. Along a similar argument, connectivity of  $\mathcal{A}(Q, \Delta)$  is also proved. We thus omit the proof of it.

Let  $\mathcal{A}^*(R)$  denote the simplicial cone over  $\mathcal{A}(R)$  with its cone point  $*$ . We then have the simplicial map

$$\pi: \mathcal{A}(R, \Delta) \rightarrow \mathcal{A}^*(R)$$

defined by forgetting  $\Delta$ , where  $\pi^{-1}(*)$  consists of all arcs in  $\mathcal{A}(R, \Delta)$  connecting two points of  $\partial_j$  for some  $j$  and homotopic relative to their end points to an arc in  $\partial_j$ . Note that  $\pi^{-1}(*)$  may be empty.

We now observe that for each arc  $u$  corresponding to a vertex of  $\pi^{-1}(\mathcal{A}(R))$ , any arc corresponding to a vertex of the fiber  $\pi^{-1}(\pi(u))$  is obtained by applying to  $u$  the twists about the component(s) of  $\partial R$  containing a point of  $\partial u$ . Let us explain this fact more precisely. We fix an orientation of  $R$ . For each  $j = 1, \dots, p$ , we put  $n_j = |\Delta^+ \cap \partial_j| = |\Delta^- \cap \partial_j|$  and set

$$\Delta^+ \cap \partial_j = \{x_1^j, \dots, x_{n_j}^j\}, \quad \Delta^- \cap \partial_j = \{y_1^j, \dots, y_{n_j}^j\}$$

so that  $x_1^j, y_1^j, x_2^j, y_2^j, \dots, x_{n_j}^j, y_{n_j}^j$  appear in this order along the orientation of  $\partial_j$  induced by that of  $R$ . We define  $t_j$  as a homeomorphism of  $R$  which is the identity outside a collar neighborhood  $N_j$  of  $\partial_j$  and satisfies the equalities

$$t_j(x_k^j) = y_k^j, \quad t_j(y_l^j) = x_{l+1}^j, \quad t_j(y_{n_j}^j) = x_1^j$$

for each  $k = 1, \dots, n_j$  and each  $l = 1, \dots, n_j - 1$ .

Pick an arc  $u$  corresponding to a vertex of  $\pi^{-1}(\mathcal{A}(R))$ . We first assume that the two points of  $\partial u$  lie in distinct components of  $\partial R$ , say  $\partial_1$  and  $\partial_2$ . We then have the equality

$$(\dagger) \quad \pi^{-1}(\pi(u)) = \{t_1^z t_2^w u \mid z, w \in \mathbb{Z}, z + w \in 2\mathbb{Z}\}.$$

We next assume that the two points of  $\partial u$  lie in the same component of  $\partial R$ , say  $\partial_1$ . Choose an arc  $u_1$  in  $\pi^{-1}(\pi(u))$  connecting  $x_1^1$  and  $y_1^1$ . We may assume that  $u_1 \cap \bar{N}_1$  consists of exactly two components  $u_+$ ,  $u_-$  with  $x_1^1 \in u_+$  and  $y_1^1 \in u_-$ , where  $\bar{N}_1$  denotes the closure of  $N_1$ . Put  $u_0 = u_1 \setminus N_1$ . For each  $k = 2, \dots, n_1$ , we define an essential simple arc  $u_k$  in  $R$  as the union

$$u_k = u_+ \cup u_0 \cup t_1^{2(k-1)} u_-.$$

The arc  $u_k$  connects  $x_1^1$  and  $y_k^1$  and belongs to  $\pi^{-1}(\pi(u))$ . The set  $\{u_1, \dots, u_{n_1}\}$  is a simplex of  $\mathcal{A}(R, \Delta)$ . We then have the equality

$$(\ddagger) \quad \pi^{-1}(\pi(u)) = \{t_1^z u_k \mid z \in \mathbb{Z}, k = 1, \dots, n_1\}.$$

The following two lemmas show that there exists an edge or a vertex of  $\mathcal{A}(R)$  whose inverse image under  $\pi$  spans a connected subcomplex of  $\mathcal{A}(R, \Delta)$ .

**Lemma 7.3.** *Assume  $g \geq 1$ . Let  $\{u, v\}$  be an edge of  $\pi^{-1}(\mathcal{A}(R))$  such that*

- *$u$  and  $v$  are non-separating in  $R$ ;*



- a single component of  $\partial R$  contains both  $\partial u$  and  $\partial v$ ; and
- when we cut  $R$  along  $u$  and obtain a connected surface  $R_u$ , the arc  $v$  connects the two boundary components of  $R_u$  containing  $u$ .

Then the full subcomplex of  $\mathcal{A}(R, \Delta)$  spanned by the union  $\pi^{-1}(\pi(u)) \cup \pi^{-1}(\pi(v))$  is connected.

*Proof.* We may assume that  $\partial u$  and  $\partial v$  are contained in  $\partial_1$ , and put  $\partial = \partial_1$ . We use the same notation right before the lemma and put  $n = n_1$  and  $t = t_1$ . We also may assume that  $\partial u = \partial v = \{x_1^1, y_1^1\}$  and that  $u$  is decomposed into three arcs,  $u = u_+ \cup u_0 \cup u_-$ , so that  $u_+$  and  $u_-$  are the two components of  $u \cap \bar{N}_1$  with  $x_1^1 \in u_+$  and  $y_1^1 \in u_-$ ; and we have  $u_0 = u \setminus N_1$ .

To prove the lemma, by equation  $(\ddagger)$ , it suffices to show that there exists a path in  $\mathcal{A}(R, \Delta)$  connecting  $u$  and  $tu$  and consisting of vertices in  $\pi^{-1}(\pi(u)) \cup \pi^{-1}(\pi(v))$ . If  $n \geq 2$ , then define an essential simple arc  $w$  in  $R$  as the union  $w = u_+ \cup u_0 \cup t^2 u_-$ . The sequence  $u, w, tu$  defines a path in  $\mathcal{A}(R, \Delta)$ . We assume  $n = 1$ . The third assumption on  $u$  and  $v$  in the lemma implies that  $u$  is disjoint from either  $tv$  or  $t^{-1}v$ . It follows that either the sequence  $u, v, tu$  or the sequence  $u, tv, tu$  defines a path in  $\mathcal{A}(R, \Delta)$ .  $\square$

**Lemma 7.4.** *Assume  $p \geq 2$ . Let  $u$  be an arc corresponding to a vertex of  $\mathcal{A}(R, \Delta)$  and connecting distinct components of  $\partial R$ . Then the full subcomplex of  $\mathcal{A}(R, \Delta)$  spanned by  $\pi^{-1}(\pi(u))$  is connected.*

*Proof.* We may assume that  $u$  connects  $\partial_1$  with  $\partial_2$ . Both  $\{u, t_2^2 u\}$  and  $\{u, t_1^{-1} t_2 u\}$  are edges of  $\mathcal{A}(R, \Delta)$ . The lemma then follows from equation  $(\dagger)$ .  $\square$

*Proof of connectivity of  $\mathcal{A}(R, \Delta)$  in the case of  $(g, p) \neq (0, 1)$ .* We note that  $\mathcal{A}(R)$  is connected by Theorem (a) in [12]. We also note that for any vertex  $x$  of  $\mathcal{A}(R, \Delta)$ , the image of the link of  $x$  in  $\mathcal{A}(R, \Delta)$  under the map  $\pi: \mathcal{A}(R, \Delta) \rightarrow \mathcal{A}^*(R)$  contains the link of  $\pi(x)$  in  $\mathcal{A}(R)$ .

Assume  $g \geq 1$  and pick an edge  $\{u, v\}$  of  $\mathcal{A}(R, \Delta)$  satisfying the condition in Lemma 7.3. Let  $w$  be a vertex of  $\mathcal{A}(R, \Delta)$ . Since  $\mathcal{A}(R)$  is connected, there exists a path in  $\mathcal{A}(R)$  joining  $\pi(w)$  to  $\pi(u)$ . The fact stated in the end of the previous paragraph implies that there exists a path in  $\mathcal{A}(R, \Delta)$  joining  $w$  to a vertex in  $\pi^{-1}(\pi(u))$ . Since any vertex in  $\pi^{-1}(\pi(u))$  is joined to  $u$  in  $\mathcal{A}(R, \Delta)$  by Lemma 7.3, connectivity of  $\mathcal{A}(R, \Delta)$  follows. If  $g = 0$  and  $p \geq 2$ , then we can prove connectivity of  $\mathcal{A}(R, \Delta)$  in a similar way, using Lemma 7.4 in place of Lemma 7.3.  $\square$

Let us make a comment on the proof of connectivity of  $\mathcal{A}(Q, \Delta)$  in the case of  $(g, p) \neq (0, 1)$ . Let  $\mathcal{A}^*(Q)$  denote the simplicial cone over  $\mathcal{A}(Q)$  with its cone point  $*$ . We then have the simplicial map  $\pi: \mathcal{A}(Q, \Delta) \rightarrow \mathcal{A}^*(Q)$  defined by forgetting  $\Delta$ . To prove connectivity of  $\mathcal{A}(Q, \Delta)$  along the proof for  $\mathcal{A}(R, \Delta)$ , we need to know connectivity of the full subcomplex of  $\mathcal{A}(Q)$  spanned by  $\pi(V_a(Q, \Delta)) \setminus \{*\}$ , in place of connectivity of  $\mathcal{A}(R)$ . The set  $\pi(V_a(Q, \Delta)) \setminus \{*\}$  is equal to the set of all vertices corresponding to arcs connecting two points of  $\partial Q \setminus \partial_0$ . Connectivity of this full subcomplex also follows from Theorem (a) in [12].

**7.2. Injections of  $\mathcal{D}(X, \partial)$  and  $\mathcal{D}(Y)$ .** Let  $X = S_{g,3}$  and  $Y = S_{g,2}$  be surfaces with  $g \geq 1$  and fix a boundary component  $\partial$  of  $X$ . In this subsection, we discuss the properties of the complexes  $\mathcal{D}(X, \partial)$  and  $\mathcal{D}(Y)$  stated in the following two lemmas and prove Proposition 7.1. Let us say that a simplicial complex  $\mathcal{E}$  is *chain-connected*

if for any two simplices  $\sigma, \tau$  of maximal dimension in  $\mathcal{E}$ , there exists a sequence of simplices of maximal dimension in  $\mathcal{E}$ ,  $\sigma_0, \sigma_1, \dots, \sigma_m$ , with  $\sigma_0 = \sigma$  and  $\sigma_m = \tau$  and with  $\sigma_j \cap \sigma_{j+1}$  a simplex of codimension one for each  $j = 0, 1, \dots, m-1$ .

**Lemma 7.5.** *In the above notation, the following assertions hold:*

- (i) *Let  $\sigma$  be a simplex of  $\mathcal{D}(X, \partial)$  of codimension one. We denote by  $X_\sigma$  the surface obtained by cutting  $X$  along arcs in  $\sigma$ . Then the number of simplices of maximal dimension in  $\mathcal{D}(X, \partial)$  containing  $\sigma$  is equal to three if the component of  $X_\sigma$  containing  $\partial$  contains exactly two arcs in  $\sigma$ , and otherwise that number is equal to four.*
- (ii) *The complex  $\mathcal{D}(X, \partial)$  is chain-connected.*

**Lemma 7.6.** *In the above notation, the following assertions hold:*

- (i) *For each simplex  $\sigma$  of  $\mathcal{D}(Y)$  of codimension one, the number of simplices of maximal dimension in  $\mathcal{D}(Y)$  containing  $\sigma$  is equal to three.*
- (ii) *The complex  $\mathcal{D}(Y)$  is chain-connected.*

Before proving these two lemmas, let us recall basic facts on punctured surfaces and ideal arcs in them, which are fully discussed in [26]. Let  $T$  be a closed surface of positive genus  $g$ , and let  $P$  be a non-empty finite subset of  $T$ . The pair  $(T, P)$  is then called a *punctured surface*. Let  $I$  denote the closed unit interval. We mean by an *ideal arc* in  $(T, P)$  is the image of a continuous map  $f: I \rightarrow T$  such that

- we have  $f(\partial I) \subset P$  and  $f(I \setminus \partial I) \subset T \setminus P$ ;
- $f$  is injective on  $I \setminus \partial I$ ; and
- there exists no closed disk  $D$  embedded in  $T$  with  $\partial D = f(I)$  and  $(D \setminus \partial D) \cap P = \emptyset$ .

Two ideal arcs  $l_1, l_2$  in  $(T, P)$  are said to be *isotopic* if we have  $l_1 \cap P = l_2 \cap P$ ; and  $l_1$  and  $l_2$  are isotopic relative to  $l_1 \cap P$  as arcs in  $(T \setminus P) \cup (l_1 \cap P)$ . We mean by an *ideal triangulation* of  $(T, P)$  is a cell division  $\delta$  of  $T$  such that

- (a) the set of 0-cells of  $\delta$  is  $P$ ;
- (b) each 1-cell of  $\delta$  is an ideal arc in  $(T, P)$ ; and
- (c) each 2-cell of  $\delta$  is a *triangle*, that is, it is obtained by attaching a Euclidean triangle  $\tau$  to the 1-skeleton of  $\delta$ , mapping each vertex of  $\tau$  to a 0-cell of  $\delta$ , and each edge of  $\tau$  to a 1-cell of  $\delta$ .

Let  $S$  be a surface of genus  $g$  with  $|P|$  boundary components. Suppose that  $T$  is obtained from  $S$  by shrinking each component of  $\partial S$  into a point and that  $P$  is the set of points into which components of  $\partial S$  are shrunk. The natural map from  $S$  onto  $T$  induces the bijection from  $V_a(S)$  onto the set of isotopy classes of ideal arcs in  $(T, P)$ . Under this identification, a simplex of  $\mathcal{A}(S)$  of maximal dimension corresponds to an ideal triangulation of  $T$ .

*Proof of Lemma 7.6 (i).* Let  $Y^*$  denote the punctured surface obtained from  $Y$  by shrinking each component of  $\partial Y$ . We identify  $V_a(Y)$  with the set of isotopy classes of ideal arcs in  $Y^*$ . Each simplex  $\sigma$  of  $\mathcal{D}(Y)$  of maximal dimension corresponds to a *squaring* of  $Y^*$ , that is, when we cut  $Y^*$  along arcs in  $\sigma$ , we obtain finitely many squares whose vertices are punctures of  $Y^*$  and edges are arcs in  $\sigma$ . We see that

- for each square  $\Pi$  of the squaring, diagonal vertices of  $\Pi$  correspond to the same puncture of  $Y^*$ , and the two vertices of each edge of  $\Pi$  correspond to distinct punctures of  $Y^*$  (see Figure 7 (a)); and

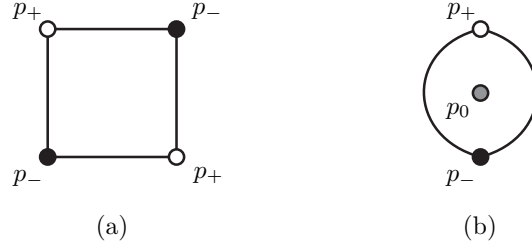


FIGURE 7. (a) A square with  $p_{\pm}$  the punctures of  $Y^*$ ; (b) A digon with  $p_{\pm}$  and  $p_0$  the punctures of  $X^*$ .

- for each arc  $l$  in  $\sigma$ , the two squares having  $l$  as an edge are distinct. In other words, for each square  $\Pi$  of the squaring, any two distinct edges of  $\Pi$  correspond to distinct arcs in  $\sigma$ .

Each simplex of  $\mathcal{D}(Y)$  of maximal dimension consists of  $4g$  arcs, and the number of squares in the squaring associated with it is equal to  $2g$ . If we cut  $Y^*$  along arcs in a simplex of  $\mathcal{D}(Y)$  of codimension one, then we obtain one hexagon  $H$  and  $2g - 2$  squares. There exist exactly three arcs in  $H$  each of which decomposes  $H$  into two squares. Lemma 7.6 (i) follows.  $\square$

*Proof of Lemma 7.5 (i).* Let  $X^*$  denote the punctured surface obtained from  $X$  by shrinking each component of  $\partial X$ . Let  $p_0$  denote the puncture of  $X^*$  corresponding to  $\partial$ . As in the proof of Lemma 7.6 (i), if we cut  $X^*$  along arcs in a simplex of  $\mathcal{D}(X, \partial)$  of maximal dimension, then we obtain  $2g$  squares and one digon containing  $p_0$  (see Figure 7 (b)). Such a simplex consists of  $4g + 1$  arcs. Let  $\tau$  be a simplex of  $\mathcal{D}(X, \partial)$  of codimension one, and cut  $X^*$  along arcs in  $\tau$ . If the component containing  $p_0$  contains exactly four edges, then the number of simplices of maximal dimension containing  $\tau$  is equal to four. Otherwise the component containing  $p_0$  contains exactly two edges, and we have one hexagon. The number of simplices of maximal dimension containing  $\tau$  is thus equal to three. Lemma 7.5 (i) follows.  $\square$

*Proof of Lemma 7.6 (ii).* We prove that  $\mathcal{D}(Y)$  is connected, using the technique due to Putman [27] to show connectivity of a simplicial complex on which  $\text{PMod}(Y)$  acts. Let  $l$  be the arc in Figure 8 (b). We pick an arc  $r$  corresponding to a vertex of  $\mathcal{D}(Y)$  and show that  $l$  and  $r$  can be connected by a path in  $\mathcal{D}(Y)$ . We define  $T$  as the set consisting of the Dehn twists about the curves in Figure 8 (a) and their inverses. It is known that  $\text{PMod}(Y)$  is generated by  $T$  (see [10]). Since  $l$  and  $r$  are sent to each other by an element of  $\text{PMod}(Y)$ , there exist elements  $h_1, \dots, h_n$  of  $T$  with  $r = h_1 \cdots h_n l$ . We note that for each  $h \in T$ , either  $hl = l$  or  $hl$  and  $l$  are disjoint. The sequence of vertices of  $\mathcal{D}(Y)$ ,

$$l, h_1 l, h_1 h_2 l, \dots, h_1 \cdots h_n l = r,$$

therefore forms a path in  $\mathcal{D}(Y)$ . Connectivity of  $\mathcal{D}(Y)$  follows.

To prove Lemma 7.6 (ii), it suffices to show that the link of each simplex of  $\mathcal{D}(Y)$  of codimension at least two is connected. The idea to derive chain-connectivity from connectivity of such a link is due to Hatcher [12] and is also used in the proof of Proposition 4.7 in [6].

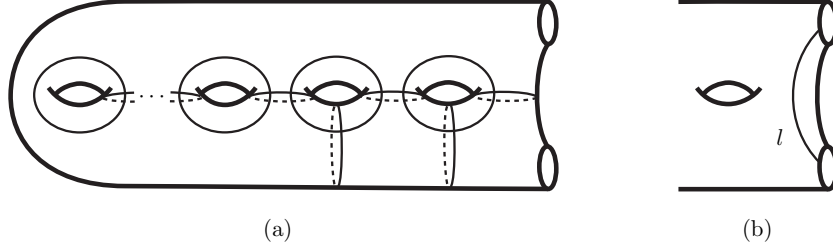


FIGURE 8. (a)  $\text{PMod}(Y)$  is generated by Dehn twists about these curves.

Let  $\sigma$  be a simplex of  $\mathcal{D}(Y)$  of codimension at least two. We identify the arc corresponding to a vertex of  $\mathcal{D}(Y)$  with an ideal arc in the punctured surface  $Y^*$ . Let  $p_+$  and  $p_-$  denote the two punctures of  $Y^*$ . We cut  $Y^*$  along arcs in  $\sigma$  and obtain finitely many surfaces  $Y_1^*, \dots, Y_m^*$ . For each  $j = 1, \dots, m$ , let  $\Delta_j^+$  and  $\Delta_j^-$  denote the sets of points of  $Y_j^*$  corresponding to  $p_+$  and  $p_-$ , respectively, and set

$$\Delta_j = \Delta_j^+ \cup \Delta_j^-.$$

For each  $j$ , the set  $\Delta_j$  with this decomposition satisfies the two conditions stated in the beginning of Section 7.1. Each vertex of the complex  $\mathcal{A}(Y_j^*, \Delta_j)$  is identified with a vertex of the link of  $\sigma$  in  $\mathcal{D}(Y)$ . If there exist distinct  $j, k = 1, \dots, m$  with both  $\mathcal{A}(Y_j^*, \Delta_j)$  and  $\mathcal{A}(Y_k^*, \Delta_k)$  non-empty, then the link of  $\sigma$  is connected. Otherwise, since  $\sigma$  is of codimension at least two, there exists a unique  $j$  with  $\mathcal{A}(Y_j^*, \Delta_j)$  of dimension at least one. By Proposition 7.2 (ii),  $\mathcal{A}(Y_j^*, \Delta_j)$  is connected.  $\square$

Lemma 7.5 (ii) can also be proved along a similar idea using Proposition 7.2 and using Figure 9 (a) in place of Figure 8 (a).

*Proof of Proposition 7.1.* We first note that each vertex of  $\mathcal{D}(X, \partial)$  (resp.  $\mathcal{D}(Y)$ ) is contained in a simplex of maximal dimension. Let  $\psi: \mathcal{D}(Y) \rightarrow \mathcal{D}(Y)$  be an injective simplicial map. Pick a simplex  $\sigma$  of  $\mathcal{D}(Y)$  of maximal dimension. Injectivity of  $\psi$  and Lemma 7.6 (i) imply that for each face  $\tau$  of  $\psi(\sigma)$  of codimension one, any simplex of  $\mathcal{D}(Y)$  of maximal dimension containing  $\tau$  is contained in the image of  $\psi$ . By Lemma 7.6 (ii), any simplex  $\mathcal{D}(Y)$  of maximal dimension is contained in the image of  $\psi$ . Surjectivity of  $\psi$  thus follows.

Let  $\phi: \mathcal{D}(X, \partial) \rightarrow \mathcal{D}(X, \partial)$  be an injective simplicial map. Fix a simplex  $\sigma$  of  $\mathcal{D}(X, \partial)$  of maximal dimension. For each face  $\tau$  of  $\sigma$  of codimension one, let  $n(\tau)$  denote the number of simplices of maximal dimension in  $\mathcal{D}(X, \partial)$  containing  $\tau$ . The proof of Lemma 7.5 (i) shows that the number of faces  $\tau$  of  $\sigma$  of codimension one with  $n(\tau) = 4$  is equal to two and that any other face  $\rho$  of  $\sigma$  of codimension one satisfies  $n(\rho) = 3$ . Injectivity of  $\phi$  implies that if  $n(\tau) = 4$ , then  $n(\phi(\tau)) = 4$ . It follows that if  $n(\tau) = 3$ , then  $n(\phi(\tau)) = 3$ . Any simplex of  $\mathcal{D}(Y)$  of maximal dimension containing  $\phi(\tau)$  is therefore contained in the image of  $\phi$ . As in the last paragraph, by Lemma 7.5 (ii), surjectivity of  $\phi$  follows.  $\square$

## 8. SURJECTIVITY OF SUPERINJECTIVE MAPS

We are now ready to show surjectivity of any superinjective map from  $\mathcal{CP}_n(S)$  into itself and any superinjective map from  $\mathcal{CP}_s(S)$  into itself.

**8.1. The case  $p = 2$ .** Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Recall that we have the simplicial map  $\pi: \mathcal{C}(S) \rightarrow \mathcal{C}^*(\bar{S})$ , where  $\bar{S}$  is the closed surface obtained from  $S$  by attaching disks to all components of  $\partial S$ . For each vertex  $v$  of  $\mathcal{C}(\bar{S})$ , we define  $\mathcal{C}(S)_v$  as the full subcomplex of  $\mathcal{C}(S)$  spanned by  $\pi^{-1}(v)$ , which is connected by Theorem 7.1 in [18]. We also have the simplicial maps

$$\theta_n: \mathcal{CP}_n(S) \rightarrow \mathcal{C}^*(\bar{S}), \quad \theta_s: \mathcal{CP}_s(S) \rightarrow \mathcal{C}^*(\bar{S})$$

associated to  $\pi$ . For a vertex  $v$  of  $\mathcal{C}(\bar{S})$ , we denote by  $\mathcal{CP}_n(S)_v$  the full subcomplex of  $\mathcal{CP}_n(S)$  spanned by  $\theta_n^{-1}(v)$ . The subcomplex  $\mathcal{CP}_s(S)_v$  of  $\mathcal{CP}_s(S)$  is also defined in the same way. The following lemma will be used in the sequel.

**Lemma 8.1.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Then*

- (i) *for each non-separating curve  $\alpha$  in  $\bar{S}$ , the complex  $\mathcal{CP}_n(S)_\alpha$  is connected.*
- (ii) *for each separating curve  $\beta$  in  $\bar{S}$ , the complex  $\mathcal{CP}_s(S)_\beta$  is connected.*

*Proof.* Pick a non-separating curve  $\alpha$  in  $\bar{S}$ . Lemma 4.5 in [22] shows that the link of each vertex of  $\mathcal{C}(S)_\alpha$  is connected. Combining this fact with connectivity of  $\mathcal{C}(S)_\alpha$ , we obtain assertion (i). Similarly, assertion (ii) is proved by using Lemma 4.10 in [22] asserting that the link of each vertex of  $\mathcal{C}(S)_\beta$  is connected for any separating curve  $\beta$  in  $\bar{S}$ .  $\square$

We define  $V_n(\bar{S})$  as the subset of  $V(\bar{S})$  consisting of non-separating curves in  $\bar{S}$  and define  $\mathcal{C}_n(\bar{S})$  as the full subcomplex of  $\mathcal{C}(\bar{S})$  spanned by  $V_n(\bar{S})$ .

We now outline the proof of surjectivity of a superinjective map  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$ . We first show that  $\phi$  sends the fiber of  $\theta_n$  over each vertex of  $V_n(\bar{S})$  onto the fiber of  $\theta_n$  of the same kind. It follows that  $\phi$  induces a map  $\bar{\phi}$  from  $V_n(\bar{S})$  into itself. We next show that  $\bar{\phi}$  defines a superinjective map from  $\mathcal{C}_n(\bar{S})$  into itself. The latter map is induced by an element of  $\text{Mod}^*(\bar{S})$  due to Irmak [15] and is thus surjective. We then conclude surjectivity of  $\phi$ .

**Theorem 8.2.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Then any superinjective map  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  is surjective.*

*Proof.* Throughout this proof, we mean by an HBP in  $S$  a non-separating HBP in  $S$ . For each vertex  $a$  of  $\mathcal{CP}_n(S)$ , we denote by  $\text{Lk}(a)$  the link of  $a$  in  $\mathcal{CP}_n(S)$ . We denote by  $\text{Lk}(a)^0$  the set of vertices in  $\text{Lk}(a)$ . Let  $V_n(S)$  denote the subset of  $V(S)$  consisting of non-separating curves in  $S$ . It follows from Lemmas 2.6 and 5.3 that we have a map  $\Phi: V_n(S) \rightarrow V_n(S)$  satisfying the equality  $\phi(\{\alpha, \beta\}) = \{\Phi(\alpha), \Phi(\beta)\}$  for any HBP  $\{\alpha, \beta\}$  in  $S$ .

In Lemma 5.2, we have shown the equality  $\phi(\text{Lk}(b)) = \text{Lk}(\phi(b))$  for each 2-HBP  $b$  in  $S$ . Behavior of  $\phi$  in the link of a 1-HBP is obtained in the following:

**Claim 8.3.** *For each 1-HBP  $a$  in  $S$  and each curve  $\alpha$  in  $a$ , we have the equality*

$$\phi(\{b \in \text{Lk}(a)^0 \mid \alpha \in b\}) = \{c \in \text{Lk}(\phi(a))^0 \mid \Phi(\alpha) \in c\}.$$

*Proof.* Let  $X$  and  $X'$  denote the components of  $S_a$  and  $S_{\phi(a)}$  of positive genus, respectively. We define  $\partial$  and  $\partial'$  as the boundary components of  $X$  and  $X'$  that correspond to  $\alpha$  and  $\Phi(\alpha)$ , respectively. We claim that  $\phi$  induces an injective simplicial map from  $\mathcal{D}(X, \partial)$  into  $\mathcal{D}(X', \partial')$ .

There is one-to-one correspondence between essential simple arcs in  $X$  connecting the two components of  $\partial X \setminus \partial$  and curves in  $X$  cutting off a pair of pants containing the two components of  $\partial X \setminus \partial$ . The pair of  $\alpha$  and such a curve in  $X$  is a 2-HBP

in  $S$ . The same property holds for  $X'$  and  $\partial'$  in place of  $X$  and  $\partial$ , respectively. Lemma 5.4 shows that  $\phi$  induces a simplicial map from  $\mathcal{D}(X, \partial)$  into  $\mathcal{D}(X', \partial')$ , which is injective because so is  $\phi$ .

By Proposition 7.1, this induced map is surjective. It follows that  $\phi$  sends the set of 2-HBPs in  $\text{Lk}(a)$  containing  $\alpha$  onto the set of 2-HBPs in  $\text{Lk}(\phi(a))$  containing  $\Phi(\alpha)$ . Let  $\alpha'$  denote the curve in  $a$  distinct from  $\alpha$ . Applying the same argument to  $\alpha'$  in place of  $\alpha$ , we see that  $\phi$  sends the set of 2-HBPs in  $\text{Lk}(a)$  containing  $\alpha'$  onto the set of 2-HBPs in  $\text{Lk}(\phi(a))$  containing  $\Phi(\alpha')$ . This is equivalent to that  $\phi$  sends the set of 1-HBPs in  $\text{Lk}(a)$  containing  $\alpha$  onto the set of 1-HBPs in  $\text{Lk}(\phi(a))$  containing  $\Phi(\alpha)$ . The claim follows.  $\square$

**Claim 8.4.** *For each HBP  $a$  in  $S$ , we have the equality*

$$\phi(\{b \in U_n(S) \mid \theta_n(b) = \theta_n(a)\}) = \{c \in U_n(S) \mid \theta_n(c) = \theta_n(\phi(a))\},$$

where  $U_n(S)$  denotes the set of vertices of  $\mathcal{CP}_n(S)$ .

*Proof.* We put  $\alpha = \theta_n(a)$  and  $\beta = \theta_n(\phi(a))$ , which are non-separating curves in  $\bar{S}$ . Since  $\phi$  preserves disjoint and equivalent HBPs and since  $\mathcal{CP}_n(S)_\alpha$  is connected by Lemma 8.1 (i),  $\phi$  induces a simplicial map from  $\mathcal{CP}_n(S)_\alpha$  into  $\mathcal{CP}_n(S)_\beta$ . Lemma 5.2 and Claim 8.3 show that this induced map sends the link of each vertex  $u$  of  $\mathcal{CP}_n(S)_\alpha$  onto the link of  $\phi(u)$ . The claim is proved.  $\square$

Claim 8.4 shows that  $\phi$  induces a map  $\bar{\phi}$  from  $V_n(\bar{S})$  into itself with  $\bar{\phi}(\theta_n(a)) = \theta_n(\phi(a))$  for each HBP  $a$  in  $S$ .

**Claim 8.5.** *The map  $\bar{\phi}: V_n(\bar{S}) \rightarrow V_n(\bar{S})$  defines a superinjective map from  $\mathcal{C}_n(\bar{S})$  into itself.*

*Proof.* For any curves  $\alpha, \beta \in V_n(\bar{S})$ , we have  $i(\alpha, \beta) = 0$  if and only if there exist  $a \in \mathcal{CP}_n(S)_\alpha$  and  $b \in \mathcal{CP}_n(S)_\beta$  with  $i(a, b) = 0$ . Let us refer this fact as (\*).

Pick two curves  $\alpha, \beta \in V_n(\bar{S})$ . If  $i(\alpha, \beta) = 0$ , then by (\*), there exist  $a \in \mathcal{CP}_n(S)_\alpha$  and  $b \in \mathcal{CP}_n(S)_\beta$  with  $i(a, b) = 0$ . Since  $\phi$  is simplicial, we have

$$i(\phi(a), \phi(b)) = 0, \quad \phi(a) \in \mathcal{CP}_n(S)_{\bar{\phi}(\alpha)}, \quad \phi(b) \in \mathcal{CP}_n(S)_{\bar{\phi}(\beta)}.$$

Applying (\*) again, we obtain  $i(\bar{\phi}(\alpha), \bar{\phi}(\beta)) = 0$ . The map  $\bar{\phi}$  is thus simplicial.

We next assume  $i(\bar{\phi}(\alpha), \bar{\phi}(\beta)) = 0$ . By (\*), there exist  $a' \in \mathcal{CP}_n(S)_{\bar{\phi}(\alpha)}$  and  $b' \in \mathcal{CP}_n(S)_{\bar{\phi}(\beta)}$  with  $i(a', b') = 0$ . Claim 8.4 shows that there exist  $a \in \mathcal{CP}_n(S)_\alpha$  and  $b \in \mathcal{CP}_n(S)_\beta$  with  $\phi(a) = a'$  and  $\phi(b) = b'$ . Since  $\phi$  is superinjective, we have  $i(a, b) = 0$ . Applying (\*) again, we obtain  $i(\alpha, \beta) = 0$ . The map  $\bar{\phi}$  is therefore superinjective.  $\square$

Thanks to Theorem 1.3 in [15],  $\bar{\phi}$  is induced by an element of  $\text{Mod}^*(\bar{S})$ . The map  $\bar{\phi}$  is thus surjective. Claim 8.4 implies that the image of the map  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  contains all HBPs in  $S$ . Since for each 2-HBP  $a$  in  $S$ ,  $\phi$  sends the link of  $a$  onto the link of  $\phi(a)$  by Lemma 5.2, we conclude that  $\phi$  is surjective.  $\square$

We turn to showing surjectivity of a superinjective map  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  when  $S = S_{g,2}$  with  $g \geq 2$ . For each surface  $R$ , we denote by  $V_s(R)$  the subset of  $V(R)$  consisting of separating curves in  $R$  and define  $\mathcal{C}_s(R)$  as the full subcomplex of  $\mathcal{C}(R)$  spanned by  $V_s(R)$ . When  $g \geq 3$ , we prove surjectivity of  $\psi$ , following the proof of Theorem 8.2 and associating to  $\psi$  a superinjective map  $\bar{\psi}$  from  $\mathcal{C}_s(\bar{S})$  into itself, which is surjective due to the first author [20].

When  $g = 2$ , we cannot follow the last part of this proof because  $\mathcal{C}_s(\bar{S})$  is of dimension zero. In this case, we directly show that the map  $\Psi: V_s(S) \rightarrow V_s(S)$  induced by  $\psi$  defines a superinjective map from  $\mathcal{C}_s(S)$  into itself. The latter map is surjective by [20], and thus so is  $\psi$ .

**Theorem 8.6.** *Let  $S = S_{g,2}$  be a surface with  $g \geq 2$ . Then any superinjective map  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  is surjective.*

*Proof.* Throughout this proof, we mean by an HBP in  $S$  a separating HBP in  $S$ . For each vertex  $a$  of  $\mathcal{CP}_s(S)$ , we denote by  $\text{Lk}(a)$  the link of  $a$  in  $\mathcal{CP}_s(S)$ . We denote by  $\text{Lk}(a)^0$  the set of vertices in  $\text{Lk}(a)$ . By Lemmas 2.6 and 6.3, we have a map  $\Psi: V_s(S) \rightarrow V_s(S)$  with  $\psi(\{\alpha, \beta\}) = \{\Psi(\alpha), \Psi(\beta)\}$  for any HBP  $\{\alpha, \beta\}$  in  $S$  and with  $\psi(\gamma) = \Psi(\gamma)$  for any HBC  $\gamma$  in  $S$ .

**Claim 8.7.** *Let  $a$  be a 1-HBP in  $S$ . We denote by  $\alpha_0$  the curve in  $a$  that does not separate the two components of  $\partial S$ , and denote by  $\alpha_1$  the curve in  $a$  distinct from  $\alpha_0$ . Then we have the equalities*

$$\begin{aligned} \psi(\{b \in \text{Lk}(a)^0 \mid \alpha_0 \in b\}) &= \{c \in \text{Lk}(\psi(a))^0 \mid \Psi(\alpha_0) \in c\}, \\ \psi(\{d \in \text{Lk}(a)^0 \mid \alpha_1 \in d\}) &= \{e \in \text{Lk}(\psi(a))^0 \mid \Psi(\alpha_1) \in e\}. \end{aligned}$$

*Proof.* Choose a curve  $\alpha_2$  in  $S$  such that the pair  $\{\alpha_1, \alpha_2\}$  is a 1-HBP in  $S$  disjoint and distinct from  $a$ . We denote by  $Y$  and  $Z$  the components of  $S_{\alpha_1}$  containing  $\alpha_0$  and  $\alpha_2$ , respectively. Similarly, we denote by  $Y'$  and  $Z'$  the components of  $S_{\Psi(\alpha_1)}$  containing  $\Psi(\alpha_0)$  and  $\Psi(\alpha_2)$ , respectively. By Lemma 6.4,  $\psi$  induces an injective simplicial map from  $\mathcal{D}(Y)$  into  $\mathcal{D}(Y')$  and an injective simplicial map from  $\mathcal{D}(Z)$  into  $\mathcal{D}(Z')$ . Comparing the dimensions of these simplicial complexes, we see that  $Y$  and  $Y'$  are homeomorphic and that  $Z$  and  $Z'$  are homeomorphic. Proposition 7.1 shows that the induced map from  $\mathcal{D}(Z)$  into  $\mathcal{D}(Z')$  is surjective. The first equality in the claim then follows.

Let  $e$  be an element in the right hand side of the second equality in the claim, which is a 1-HBP in  $S$ . We denote by  $e^1$  the curve in  $e$  distinct from  $\Psi(\alpha_1)$ . Since the 2-HBP  $\{\Psi(\alpha_0), e^1\}$  belongs to the right hand side of the first equality in the claim, there exists a curve  $d^1$  in  $S$  with  $\{\alpha_0, d^1\} \in \text{Lk}(a)^0$  and  $\psi(\{\alpha_0, d^1\}) = \{\Psi(\alpha_0), e^1\}$ . We then have the equality  $e = \{\Psi(\alpha_1), e^1\} = \psi(\{\alpha_1, d^1\})$  by Lemma 2.6. The second equality in the claim follows.  $\square$

Recall that we have the simplicial map  $\theta_s: \mathcal{CP}_s(S) \rightarrow \mathcal{C}^*(\bar{S})$  associated with the inclusion of  $S$  into  $\bar{S}$ . The proof of the following claim is obtained as a verbatim translation of that of Claim 8.4, by exchanging symbols appropriately and using Lemma 6.2 and Claim 8.7 in place of Lemma 5.2 and Claim 8.3.

**Claim 8.8.** *For each HBP  $a$  in  $S$ , we have the equality*

$$\psi(\{b \in U_s(S) \mid \theta_s(b) = \theta_s(a)\}) = \{c \in U_s(S) \mid \theta_s(c) = \theta_s(\psi(a))\},$$

where  $U_s(S)$  denotes the set of vertices of  $\mathcal{CP}_s(S)$ .

We now obtain the map  $\bar{\psi}: V_s(\bar{S}) \rightarrow V_s(\bar{S})$  with the equality  $\bar{\psi}(\theta_s(a)) = \theta_s(\psi(a))$  for each HBP  $a$  in  $S$ . The following claim can also be verified along an argument of the same kind as in the proof of Claim 8.5.

**Claim 8.9.** *The map  $\bar{\psi}: V_s(\bar{S}) \rightarrow V_s(\bar{S})$  defines a superinjective map from  $\mathcal{C}_s(\bar{S})$  into itself.*

If  $g \geq 3$ , then  $\bar{\psi}$  is surjective by Theorem 1.1 of [20], and surjectivity of  $\psi$  is shown along the end of the proof of Theorem 8.2. In the rest of the proof of Theorem 8.6, we assume  $g = 2$  and show that the map  $\Psi: V_s(S) \rightarrow V_s(S)$  defines a superinjective map from  $\mathcal{C}_s(S)$  into itself.

**Claim 8.10.** *Assume  $g = 2$ . For each separating curve  $\alpha$  in  $S$  which is not an HBC in  $S$ , we have the equalities*

$$\begin{aligned} \psi(\{b \in U_s(S) \mid i(\alpha, b) = 0\}) &= \{c \in U_s(S) \mid i(\Psi(\alpha), c) = 0\}, \\ \psi(\{b \in V_{sp}(S) \mid \alpha \in b\}) &= \{c \in V_{sp}(S) \mid \Psi(\alpha) \in c\}. \end{aligned}$$

*Proof.* We first assume that  $\alpha$  does not separate the two components of  $\partial S$ . Let  $Q$  and  $Q'$  denote the components of  $S_\alpha$  and  $S_{\Psi(\alpha)}$  containing  $\partial S$ , respectively, which are homeomorphic to  $S_{1,3}$ . For each separating curve  $\beta$  in  $Q$ , either  $\beta$  corresponds to an HBC in  $S$  or the pair  $\{\alpha, \beta\}$  is an HBP in  $S$ . The same property holds for  $Q'$  and  $\Psi(\alpha)$  in place of  $Q$  and  $\alpha$ , respectively. It follows that  $\psi$  induces a superinjective map from  $\mathcal{C}_s(Q)$  into  $\mathcal{C}_s(Q')$ , which is surjective by Theorem 1.1 in [20]. The first equality in the claim is proved. If  $b = \{b^1, b^2\}$  is an HBP in  $S$  with  $i(\alpha, b) = 0$  and  $\alpha \notin b$ , then any two of  $\alpha, b^1$  and  $b^2$  form an HBP in  $S$ , and we have  $\Psi(\alpha) \notin \psi(b)$  by Lemma 2.6. The second equality in the claim thus holds.

We next assume that  $\alpha$  separates the two components of  $\partial S$ . Let us denote by  $R_1$  and  $R_2$  the components of  $S_\alpha$ , and pick a curve  $\beta$  in  $R_2$  with  $\{\alpha, \beta\}$  a 1-HBP in  $S$ . We define  $R'_1$  and  $R'_2$  the components of  $S_{\Psi(\alpha)}$  with  $\Psi(\beta) \in V(R'_2)$ . By Lemma 6.4, for any two vertices  $l_1, l_2$  of  $\mathcal{D}(R_1)$ , they form an edge of  $\mathcal{D}(R_1)$  if and only if there exists a hexagon in  $\mathcal{CP}_s(S)$  satisfying the assumption in Proposition 6.1 and containing the HBPs  $\{\beta, \alpha\}$ ,  $\{\beta, \gamma_1\}$  and  $\{\beta, \gamma_2\}$ , where  $\gamma_1$  and  $\gamma_2$  are HBCs in  $R_1$  defined by the arcs  $l_1$  and  $l_2$ , respectively. The same property holds for  $R'_1$ ,  $\Psi(\alpha)$  and  $\Psi(\beta)$  in place of  $R_1$ ,  $\alpha$  and  $\beta$ , respectively. The map  $\psi$  induces an injective simplicial map from  $\mathcal{D}(R_1)$  into  $\mathcal{D}(R'_1)$ , which is surjective by Proposition 7.1. Similarly,  $\psi$  induces a simplicial isomorphism from  $\mathcal{D}(R_2)$  onto  $\mathcal{D}(R'_2)$ . The first equality in the claim is thus proved. If  $b$  is an HBP in  $S$  with  $i(\alpha, b) = 0$  and  $\alpha \notin b$ , then  $b$  is a 2-HBP in  $S$ . Since  $\Psi(\alpha)$  separates the two components of  $\partial S$ , we have  $\Psi(\alpha) \notin \psi(b)$ . The second equality in the claim follows.  $\square$

We now prove that  $\Psi: V_s(S) \rightarrow V_s(S)$  defines a superinjective map from  $\mathcal{C}_s(S)$  into itself. We first show that  $\Psi$  defines a simplicial map from  $\mathcal{C}_s(S)$  into itself. Let  $\alpha$  and  $\beta$  be separating curves in  $S$  with  $\alpha \neq \beta$  and  $i(\alpha, \beta) = 0$ . If  $\alpha$  is an HBC, then there exists a curve  $\beta'$  in  $S$  such that  $\{\beta, \beta'\}$  is an HBP disjoint from  $\alpha$ . Since  $\psi(\alpha)$  and  $\psi(\{\beta, \beta'\})$  are disjoint, we have  $i(\Psi(\alpha), \Psi(\beta)) = 0$ . If neither  $\alpha$  nor  $\beta$  is an HBC, then  $\{\alpha, \beta\}$  is an HBP because the genus of  $S$  is equal to two. We thus obtain  $i(\Psi(\alpha), \Psi(\beta)) = 0$ .

We next pick two separating curves  $\alpha, \beta$  in  $S$  with  $i(\Psi(\alpha), \Psi(\beta)) = 0$  and show  $i(\alpha, \beta) = 0$ . If both  $\alpha$  and  $\beta$  are HBCs in  $S$ , then we have  $\Psi(\alpha) = \psi(\alpha)$  and  $\Psi(\beta) = \psi(\beta)$ , and thus  $i(\alpha, \beta) = 0$  by superinjectivity of  $\psi$ .

Suppose that  $\alpha$  is not an HBC and that  $\beta$  is an HBC. Since  $\Psi(\beta)$  is an HBC and is disjoint from  $\Psi(\alpha)$ , the curve  $\Psi(\alpha)$  does not separate the two components of  $\partial S$ . It follows that  $\alpha$  does not separate the two components of  $\partial S$ . Choose a 2-HBP  $a'$  in  $S$  with  $\Psi(\alpha) \in a'$  and  $i(a', \Psi(\beta)) = 0$ . Claim 8.10 shows that there exists an HBP  $a$  in  $S$  with  $\alpha \in a$  and  $\psi(a) = a'$ . Since we have  $i(\psi(a), \psi(\beta)) = i(a', \Psi(\beta)) = 0$ , superinjectivity of  $\psi$  implies  $i(a, \beta) = 0$  and thus  $i(\alpha, \beta) = 0$ .



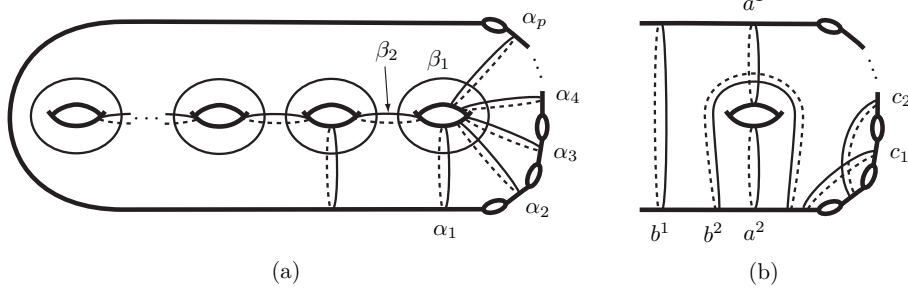


FIGURE 9. (a)  $\text{PMod}(S)$  is generated by Dehn twists about these curves. (b)  $\{a^1, a^2\}$  is a non-separating  $p$ -HBP, and  $\{b^1, b^2\}$  is a separating  $p$ -HBP.

We finally suppose that neither  $\alpha$  nor  $\beta$  is an HBC. Neither  $\Psi(\alpha)$  nor  $\Psi(\beta)$  is an HBC. If  $\Psi(\alpha) = \Psi(\beta)$ , then Claim 8.10 and injectivity of  $\psi$  imply the equality

$$\{b \in U_s(S) \mid i(b, \alpha) = 0\} = \{c \in U_s(S) \mid i(c, \beta) = 0\}.$$

We thus have the equality  $\alpha = \beta$  and particularly  $i(\alpha, \beta) = 0$ . If  $\Psi(\alpha) \neq \Psi(\beta)$ , then  $\{\Psi(\alpha), \Psi(\beta)\}$  is an HBP in  $S$  since the genus of  $S$  is equal to two. By Claim 8.10, there exist curves  $\alpha_1, \beta_1$  in  $S$  such that the pairs  $\{\alpha, \alpha_1\}$  and  $\{\beta, \beta_1\}$  are HBPs in  $S$  and we have the equality  $\psi(\{\alpha, \alpha_1\}) = \psi(\{\beta, \beta_1\}) = \{\Psi(\alpha), \Psi(\beta)\}$ . Injectivity of  $\psi$  implies the equality  $\{\alpha, \alpha_1\} = \{\beta, \beta_1\}$  and particularly  $i(\alpha, \beta) = 0$ .

We thus proved that  $\Psi$  defines a superinjective map from  $\mathcal{C}_s(S)$  into itself, which is surjective by Theorem 1.1 in [20]. It follows that  $\psi$  is surjective.  $\square$

**8.2. The case  $p \geq 3$ .** We first prove the following:

**Proposition 8.11.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then the following assertions hold:*

- (i) *The full subcomplex of  $\mathcal{CP}_n(S)$  spanned by all vertices corresponding to 2-HBCs and  $p$ -HBPs is connected.*
- (ii) *If  $p \geq 3$ , then the full subcomplex of  $\mathcal{CP}_s(S)$  spanned by all vertices corresponding to 2-HBCs and  $p$ -HBPs is connected.*

*Proof.* We follow the idea in Lemma 2.1 of [27] to prove connectivity of simplicial complexes on which  $\text{PMod}(S)$  acts, as in the proof of Lemma 7.6 (ii). It is known that  $\text{PMod}(S)$  is generated by Dehn twists about the curves described in Figure 9 (a) (see [10]). Let  $a = \{a^1, a^2\}$  denote the non-separating  $p$ -HBP in Figure 9 (b). For any 2-HBC  $\alpha$  in  $S$ , there exists  $h \in \text{PMod}(S)$  such that  $\alpha$  is disjoint from  $ha$ . To prove assertion (i), it thus suffices to show that for each curve  $\gamma$  in Figure 9 (a),  $t_\gamma a$  and  $a$  can be connected by a path in  $\mathcal{CP}_n(S)$  consisting of vertices corresponding to  $p$ -HBPs and 2-HBCs. This is true because  $t_{\beta_1} a$  and  $a$  can be connected via the 2-HBC  $c_1$  in Figure 9 (b). Assertion (i) follows.

Let  $b = \{b^1, b^2\}$  denote the separating  $p$ -HBP in Figure 9 (b). For any 2-HBC  $\alpha$  in  $S$ , there exists  $h \in \text{PMod}(S)$  such that  $\alpha$  is disjoint from  $hb$ . For any separating  $p$ -HBP  $b'$  in  $S$ , there exists  $h' \in \text{PMod}(S)$  such that  $b'$  is disjoint from  $h'c_1$ . Since  $b$  is disjoint from  $c_1$ , we have the path  $b', h'c_1, h'b$  in  $\mathcal{CP}_s(S)$ . To prove assertion (ii), it thus suffices to show that for each curve  $\gamma$  in Figure 9 (a),  $t_\gamma b$  and  $b$  can

be connected by a path in  $\mathcal{CP}_s(S)$  consisting of vertices corresponding to  $p$ -HBPs and 2-HBCs. This is true because  $t_{\alpha_2}b$  and  $b$  can be connected via the 2-HBC  $c_2$  in Figure 9 (b); for each  $j = 3, \dots, p$ ,  $t_{\alpha_j}b$  and  $b$  can be connected via  $c_1$ ; and  $t_{\beta_2}b$  and  $b$  can be connected via  $c_1$ .  $\square$

**Theorem 8.12.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 3$ , and let  $\phi: \mathcal{CP}_n(S) \rightarrow \mathcal{CP}_n(S)$  and  $\psi: \mathcal{CP}_s(S) \rightarrow \mathcal{CP}_s(S)$  be superinjective maps. Then  $\phi$  and  $\psi$  are surjective.*

*Proof.* In Lemmas 3.6 and 3.7, we have already shown that for each  $j$  and each  $k$ , the map  $\phi$  preserves  $j$ -HBPs and  $k$ -HBCs, respectively. We prove surjectivity of  $\phi$  by induction on  $p$ . For each 2-HBC  $\alpha$  in  $S$ ,  $\phi$  induces a map between the links of  $\alpha$  and  $\phi(\alpha)$  in  $\mathcal{CP}_n(S)$ . This induced map can be identified with a superinjective map from  $\mathcal{CP}_n(S_{g,p-1})$  into itself, which is surjective by the hypothesis of the induction. The image of  $\phi$  thus contains all vertices of the link of  $\phi(\alpha)$ . For each  $p$ -HBP  $b$  in  $S$ ,  $\phi$  induces a map between the links of  $b$  and  $\phi(b)$  in  $\mathcal{CP}_n(S)$ . For each curve  $\beta$  in  $b$ , the restriction of this induced map to the set of HBCs and HBPs containing  $\beta$  can be identified with a superinjective map from  $\mathcal{C}(S_{0,p+2})$  into itself. Since the latter map is surjective by Theorem 2 in [3], the image of  $\phi$  contains all vertices of the link of  $\phi(b)$ . By Proposition 8.11 (i),  $\phi$  is surjective.

Similarly, we can prove surjectivity of  $\psi$  by using Proposition 8.11 (ii).  $\square$

**Corollary 8.13.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then any superinjective map from  $\mathcal{CP}(S)$  into itself is surjective.*

*Proof.* Let  $\phi: \mathcal{CP}(S) \rightarrow \mathcal{CP}(S)$  be a superinjective map. Lemma 3.10 shows the inclusion  $\phi(\mathcal{CP}_n(S)) \subset \mathcal{CP}_n(S)$ . By Theorems 8.2 and 8.12, we have  $\phi(\mathcal{CP}_n(S)) = \mathcal{CP}_n(S)$ . Since  $\phi$  is injective, the inclusion  $\phi(\mathcal{CP}_s(S)) \subset \mathcal{CP}_s(S)$  holds. Theorems 8.6 and 8.12 show the equality  $\phi(\mathcal{CP}_s(S)) = \mathcal{CP}_s(S)$ .  $\square$

**8.3. Superinjective maps from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$ .** Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ , and let  $\phi$  be an automorphism of  $\mathcal{CP}_n(S)$ . We denote by  $\mathcal{C}_{nc}(S)$  the full subcomplex of  $\mathcal{C}(S)$  spanned by all vertices corresponding to either a non-separating curve in  $S$  or an HBC in  $S$ . Using Lemmas 3.9 and 5.3 and following a part of the proof of Theorem 6.1 in [22], we obtain a simplicial automorphism  $\Phi$  of  $\mathcal{C}_{nc}(S)$  with  $\phi(\{\alpha, \beta\}) = \{\Phi(\alpha), \Phi(\beta)\}$  for any non-separating HBP  $\{\alpha, \beta\}$  in  $S$  and with  $\phi(\gamma) = \Phi(\gamma)$  for any HBC  $\gamma$  in  $S$ . As asserted in Remark in p.102 in [15],  $\Phi$  extends to a simplicial automorphism of  $\mathcal{C}(S)$  and is thus induced by an element of  $\text{Mod}^*(S)$  by Theorem 1 in [17]. We proved the following:

**Theorem 8.14.** *Let  $S = S_{g,p}$  be a surface with  $g \geq 2$  and  $p \geq 2$ . Then any automorphism of  $\mathcal{CP}_n(S)$  is induced by an element of  $\text{Mod}^*(S)$ .*

Combining Lemma 3.10 and Theorems 8.2 and 8.12, we obtain the following:

**Corollary 8.15.** *Let  $S$  be the surface in Theorem 8.14. Then any superinjective map from  $\mathcal{CP}_n(S)$  into  $\mathcal{CP}(S)$  is induced by an element of  $\text{Mod}^*(S)$ .*

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